The Effects of Perturbations on System Dynamics Models

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Abstract

The reasons for being interested in the effects of perturbations on System Dynamics models are discussed. Various important types of perturbation are indicated. A first order model of the effects of system perturbations is then discussed and its implications considered.

The possibility of classifying models by their response to perturbations is then considered in the light of the guidance they may give as to the utility of the model.

Introduction

System Dynamics models are essentially deterministic continuous simulation models that attempt to model socioeconomic systems as non-linear control systems (Forrester, 1961). In practice because of the uncertainties attendant on such modelling it is usual to test the model with various ad hoc perturbations, e.g. input noise, parameter changes to test the 'robustness' of the model (Coyle, 1974). Published literature suggests that this type of testing which relies heavily on the experience and intuition of the modeller is by no means satisfactory (Boyle 1973, Salerno 1973, Sharp 1974).

It therefore seems worthwhile to consider the general problem of perturbations in System Dynamics models in order to

a) examine how different types of perturbation affect S.D. models.

b) to formulate methods of computing the effects of perturbations.

c) to suggest guidelines for assessing model 'robustness', since the justification of results derived from any particular model always seems in practice to rely heavily on this concept.

Types of Perturbation

A number of different types of perturbation are of interest in assessing the 'robustness' of a model. These are:-

i). Changes in initial conditions.

Initial conditions are rarely known with accuracy and therefore it is often of interest to know the effects on system performance of specific changes in initial conditions or of stochastic uncertainty in them.

ii). Changes in System Parameters.

Model parameters whether obtained by statistical methods or by the more usual process of interview are always subject to uncertainty and it is therefore of interest to know the effects of specified changes or stochastic uncertainty in parameters.

Furthermore it is usual in formulating model equations to ignore the effects of certain variables that are felt to be unimportant. This can be viewed as setting certain model parameters that are possibly non-zero equal to zero (Sharp 1974).

iii). Stochastic Effects.

Besides the stochastic uncertainties already mentioned we may be interested in the effects of introducing noise terms into individual equations. Such terms may be considered as representing either measurement error or structural error, the latter generally being more important (Malinvaud, 1970).

iv). Aggregation Error.

System Dynamics models are necessarily highly aggregated. The process of aggregation however can introduce errors in that the results are derived from an aggregate model can differ from those that would be obtained from aggregating the results of models from individual subsystems. This form of error can be regarded as equivalent to the effects of a certain perturbation on the aggregate model equations (Sharp, 1974).

Estimating the Effects of Perturbations.

The perturbations described above are either isolated (initial condition errors) or continuously acting (errors in parameters, etc.) In theory they could all be treated by applying Lyapunov methods to the relevant perturbation equation for arbitrarily large perturbations. Since S.D. models are however non-linear and no general methods exist for determining Lyapunov functions for such systems this approach - though it provides an interesting conceptual framework - does not appear likely to provide any usable techniques for estimating the effects of perturbations.

The obvious method of obtaining an estimate of the effect of perturbations is to use a first order approximation method. The application of such methods in numerical analysis has been discussed by Henrici 1963 and in control system design by among others Tomović and Vukobratović 1972. Their application to System Dynamics models is discussed in (Sharp, 1974) and (Burns and Malone, 1974).

By virtue of their special structure System Dynamics models (c.f. Sharp, 1974) can be put in the form

$$\underset{\sim}{x}_{n+1} = H(\underset{\sim}{x}_n, \underset{\sim}{\lambda}) \qquad (1)$$

with the initial state $\underset{\sim}{x}_o$ specified where $\underset{\sim}{\lambda}$ is a vector of model parameters.

Consider the system

$$\underset{\sim}{y}_{n+1} = H(\underset{\sim}{y}_n, \underset{\sim}{\lambda}) + \underset{\sim}{p}_n \tag{2}$$

with initial state $\underset{\sim}{y}_o$

where $\underset{\sim}{p}_n$ is some perturbation.

In what follows we assume that along the trajectory $\underset{\sim}{x}$ the function H is every where differentiable. This assumption is not an important restriction and simplifies the analysis. If we write

$$\underset{\sim}{\psi}_n = \underset{\sim}{y}_n - \underset{\sim}{x}_n \tag{3}$$

then from (1) and (2) we have to first order

$$\underset{\sim}{\psi}_{n+1} = J_n \underset{\sim}{\psi}_n + \underset{\sim}{p}_n \tag{4}$$

where J_n is the matrix $\left(\frac{\partial H_i}{\partial x_j}\right)_{\underset{\sim}{x}=\underset{\sim}{x}_n}$

Solution of (4) leads to

$$\underset{\sim}{\psi}_{n+1} = \prod_{r=o}^{n} J_r \underset{\sim}{\psi}_o + \sum_{q=1}^{n}\left\{\prod_{r=q}^{n} J_r\right\}\underset{\sim}{p}_{q-1} + \underset{\sim}{p}_n \tag{5}$$

Depending on the type of perturbation in which we are interested $\underset{\sim}{p}_n$ takes on different forms e.g. for the case of an error in parameters

$$\text{i.e.}\quad \underset{\sim}{y}_{n+1} = H(\underset{\sim}{y}_n, \underset{\sim}{\lambda} + \underset{\sim}{\delta\lambda}) \tag{6}$$

we have to first order that

$$\underset{\sim}{p}_n = K_n \underset{\sim}{\delta\lambda} \tag{7}$$

where $K_n = \left(\frac{\partial H_i}{\partial \lambda_k}\right)_{\underset{\sim}{x}=\underset{\sim}{x}_n}$

whilst in the case of stochastic errors

we have $\underset{\sim}{p}_n = \underset{\sim}{\varepsilon}_n$ (8)

where $\underset{\sim}{\varepsilon}_n$ is some random variable.

If suitable assumptions are made about the correlation properties of $\underset{\sim}{\varepsilon}_n$, the covariance matrix of $\underset{\sim}{\psi}_n$ is easily determined (Sharp, 1974) in the case of aggregation error the expression for $\underset{\sim}{p}_n$ is more complex in form and depends on the evolution of the subsystems of the aggregate model which is of course, ususally unknown. It follows then that even to first order approximations are needed to make possible the estimation of the aggregation error (Sharp, 1974).

The First Order Estimates

Equations (4) and (5) are of course only strictly valid for infinitesimal perturbations. They do however, give a guide as to the effects of perturbations on the model and perhaps more importantly, they show that apparently different perturbations can be treated within the same general framework rather than by a series of ad hoc modifications to the model as in the conventional approach.

It seems plausible that where analysis on the basis of first order analysis the value of ψ_n is small that the system may be said to be 'robust'. The first order analysis should then provide a basis for assessing whether this important design objective is met.

Examination of (5) shows that the effects of perturbations are propagated via the matrix products $\prod_{q}^{n} J_r$ and it therefore is of interest to enquire whether bounds for ψ_n can be deduced via bounds for these matrix products. One possible approach is via matrix norms in particular the logarithmic norm (Brauer, 1967) which seems especially suited to the matrices that arise from S.D. models. In most cases however, it would seem likely that the bounds thus derived are likely to be far too coarse to be useful. One interesting possibility is that certain bounds for matrix products given by Ostrowski, 1966 might allow the construction of more useful bounds.

Computation

Since the first order theory locates the calculation of the effects of different perturbations in a general framework it offers the possibility of reducing the computational time required to assess the effects of different types of perturbation on the model as well as providing a systematic way of deriving the necessary estimates. As can be seen from equation (5) the method requires the calculation of the sequence of matrix products $\prod_{q}^{n} J_r$ which is easily done iteratively. Alternatively an iterative method based on equation (4) can be used. Clearly the use of equation (5) to determine the effects of parameter errors enables the calculation of the effects of initial value errors to be carried out with little additional effort whereas with conventional methods that deal with the different errors separately this is not the case.

In the case of stochastic errors the same matrix products arise so the first order approach allows considerable economies of computation if various different types of errors are dealt with simultaneously. In addition approximations appear to be possible that either reduce the number of matrix products to be calculated or enable them to be calculated approximately with much less computation (Sharp 1974). Such reductions in computational requirements may in their turn make it more feasible to use, for example, optimization programs in conjunction with S.D. models.

The effects of Perturbations on Models

As remarked above the propagation of perturbations is to first order dependent on the matrix products $\prod_{q}^{n} J_r$. It is possible for fixed q and increasing n to envisage 2 extreme possibilities.

(1) as n increases $\prod_{q}^{n} J_r \, \underline{p}_{q-1}$ tends to zero so that $\underline{\psi}_n$ remains bounded.

(2) as n increases $\prod_{q}^{n} J_r \, \underline{p}_{q-1}$ grows exponentially.

Since we are dealing with nonlinear systems other types of behaviour are of course possible.

In case (1) to first order at least the effects of perturbations on the model remain bounded with time and it seems plausible that the effects of the various types of error at least if sufficiently small are unimportant. Thus we may expect that for specified inputs at least the model is a relatively good predictor. Thus in the case of specified parameter errors we might for a particular variable find a situation as depicted in fig. 1.

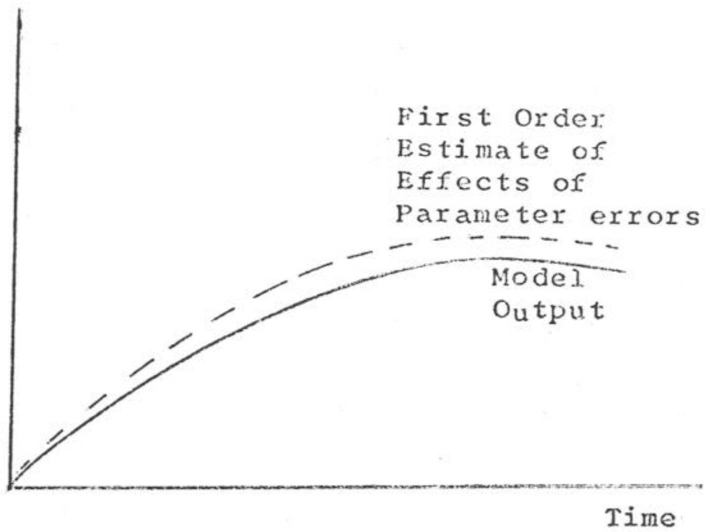

Fig. 1

For case (2) on the other hand, we would expect the model to be a poor predictor over long time spans because of the exponential build up of perturbations. In practice of course, this build up would eventually be halted by the effects of nonlinearities but before that time substantial errors may have already arisen. Thus for specified parameter errors in this case the error build up might be as in fig. 2.

Fig. 2

For this second case first order analysis is clearly inadequate for calculating actual bounds, though it could obviously be useful for indicating that problems were likely to arise when perturbations were considered. It is clear, however, that for systems of this type the connexion between the model behaviour and system behaviour may beyond a certain time horizon (say the point A in fig. 2) be very tenuous and thus the interpretation of model output may have to be very different to that applied in case (1).

It is, perhaps, interesting from this point of view to consider the 'World Dynamics' controversy. For models of type (2) we may expect from equation (5) that if the model is sensitive to errors in one parameter it is likely to be sensitive to errors in other parameters and also in initial conditions. This is certainly true for World Dynamics model as evidenced by (Burns and Malone, 1974), (Salerno, 1973), (Boyle, 1973) and (Cole and Curnow, 1973).

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