A Distributed Parameter Simulation Model of Population Growth

by

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Abstract

The techniques of Demography are well developed but their application in simulation modelling has been somewhat restricted. The classical techniques were developed in an age of manual computation and the direct application of these techniques to dynamic simulation models is somewhat cumbersome. This paper sets out the basic dynamics of population systems, its simulation and the display of the vast quantity of simulation output in such a manner that the behaviour of such systems can be easily comprehended.

Introduction

Most national economic plans, dynamic socio-economic models contain some form of population model. The simplest models assume exponential growth, while more complex models may incorporate the effects of changes in birth rates and death rates over the time span of the model. These models in general, differ from formal demographic models developed by Lotka (1907-1948) and Leslie (1945) which are in common use among demographers. These differences in structure are caused by the differing interests of the two groups each interested in one or other half of the feedback loop causing interactions between size and structure of a population with that of the related economy.

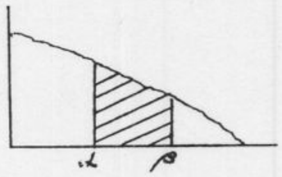
The model described below was built as part of a larger socio-economic model of Sri-lanka and the data used in the examples relate to this country. The simulations presented are not projections but illustrative examples driven in such a manner to excite specific dynamics in the models.

The difficulty with classical dermographic techniques is that they are formulated in terms of coefficient matricies estimated from census data and it is difficult to modify these coefficients in relation to external influences unless extremely simple assumptions are made of the future. A different approach to the problem is to model the population system as a distributed parameter feedback system and to modify the feedback and attenuation functions by external influences.

The Distributed Parameter Model

If we consider an age specific population as if in a pipeline shown in fig. 1. We have newly born infants entering at A and the whole population drifting towards B at the same rate as time (as age is elapsed time). The population in the pipeline is attenuated by deaths in a non-linear manner where the leakage rate depends on age. There is feedback from the female population in the age range 15-55 to the newly born.

The system can be mathematically formulated if we define a continuous variable in age and time C(age,time) where



$$P_{\alpha}^{\beta}(t) = \int_{\alpha}^{\beta} C(a,t)da \qquad (1)$$

where $P^{B}(t)$ is the population between the ages α and β at time t.

A non-linear function Da,t) giving the probability of death of an individual of age a at time t over a unit interval of time could be defined such that the death rate Dr(t) within the whole population is given by

$$Dr(t) = \int_{0}^{100} D(a,t).C(a,t)da \qquad (2)$$

The probability of birth to women of age a at time t could be defined as a maternal age specific fertility function F(a,t) such that the birth rate Br(t) to all women at time t is given by

$$Br(t) = \int_{15}^{55} F(a,t).Cf(a,t)da \qquad (3)$$

where Cf is the age specific female population distribution at time t.

AndC is a function of age and time we have

$$dC = \frac{\partial C}{\partial a}.da + \frac{\partial C}{\partial t} dt \qquad (4)$$

but $\quad da = dt \qquad (5)$

as age is elapsed time

Consider a group born some instant in the past. If we assume no migration the change in this group can be only due to death.

$$dC = -C(a,t).D(a,t).dt \qquad (6)$$

substituting (5) and (6) in (4) we have

$$\left(\frac{\partial C}{\partial t}\right) = -C.D - \left(\frac{\partial C}{\partial a}\right)_t \qquad (7)$$

The first term accounts for deaths and the 2nd for effects of aging.

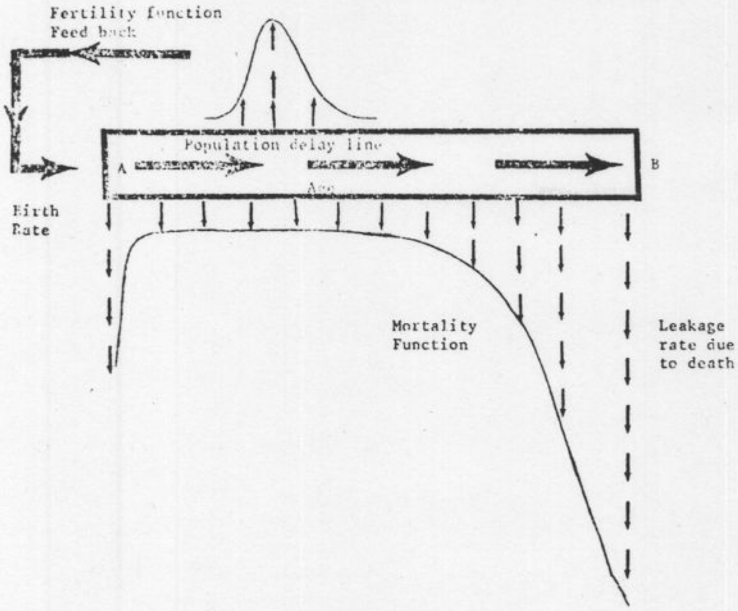


Fig. 1

Fig.1 Distributed parameter model. Fertility & mortality functions influenced by external forces and their defining parameters are continuous variables in time

Fig.2 Age specific population. No. of persons at a given age. Census data for Sri-Lenka 1967

Fig.3 Age specific mortality function. The probability of Death over a year at each age. Double exponential analitic function used in simulation

Fig. 2

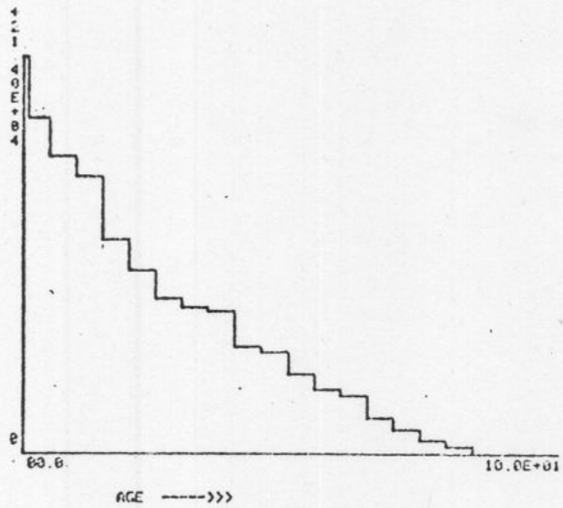
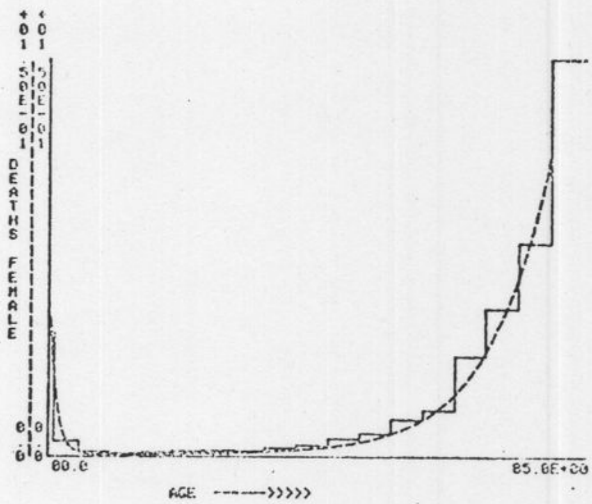


Fig. 3



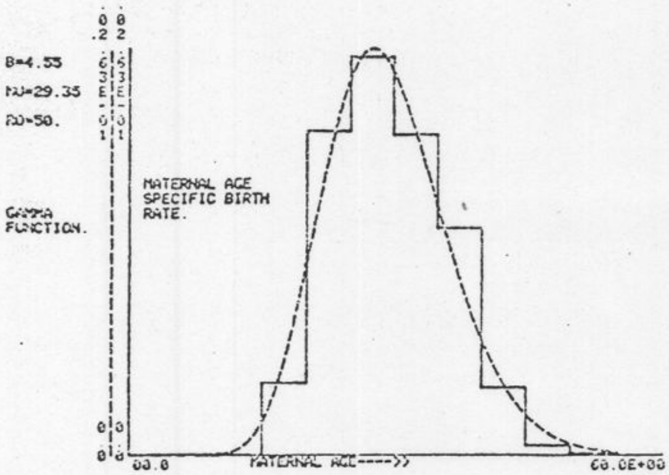


Fig. 4(a)

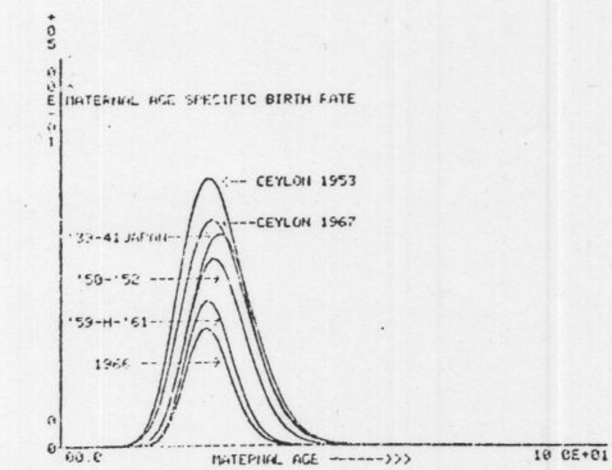


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Fig. 4(b)

Maternal age specific fertility function. Probability of birth over one year at each maternal age and Gamma function used during simulation.

Changes in fertility function with fall in total fertility.

Equations (2) (3) and (7) describe the population system which can be numerically evaluated given the initial conditions of the age specific population and the time paths of the fertility and mortality functions. The sampled value of C, F and D are readily available from census data and the initial conditions of the age specific population can be easily set up.

Simulations of Population Growth and Migration

To link such a population model to larger models we must reduce the fertility and mortality functions to some analytic form so that some meaningful parameters with which the influences of the external system could be transfered. In the simulation shown in Fig. 3 and 4 the author has used a gamma distribution to approximate the fertility function and a double exponential function for the mortality function.

The simulation results shown in Fig 5,6 and 7 show the formation of population waves in the system under different assumptions changes in the size of the fertility function.

The Relationship between GNP and Mortality and Fertility

The problems of linking dermographic models to economic development models is that there is no obvious theory to rely on. It is the current fashion to relate the growth of per capita income to declining fertility. This is generally derived on an emperical basis even though it is apparent from Fig. 8 that this is by no means a clear cut relation. The relation between the gross national product and life expectancy (the age at which $\int_0^{LE} D(a,t)da = 1$) is far more apparent, as seen in Fig. 9.

In both the mortality and fertility the author could not use simple relationships with GNP per capita as it was quite apparent that Sri-lanka was anomolous and lay well outside any emperical curve that could be fitted through the international data set. If it is accepted that there is a relationship between GNP per capita and life expectancy then this anomoly would have to be explored in terms of distribution of incomes within the country being more equal than in the other underdeveloped countries, but this would make the underdeveloped communist countries anomolous in being close to the curve.

A different approach to the problem is to make the assumption that the total fertility rate of a society falls approximately a generation after the fall in infant mortality (Coale) and that this is the dominant factor and economic forces are less dominant. Under this assumption during the transient phase from high fertility rates to low fertility rates the generally accepted family size is continuously changing. As a result mothers who entered their family forming period in an age of large families would find that the socially accepted family size would have already been reached fairly early in life than planned at the outset, as the accepted family size is falling. The consequence of this and the smaller family size is that the fertility function skews towards younger maternal age. This can be allowed for in simulation by changing the three parameters of the fertility function in a related manner as shown in fig. 10.

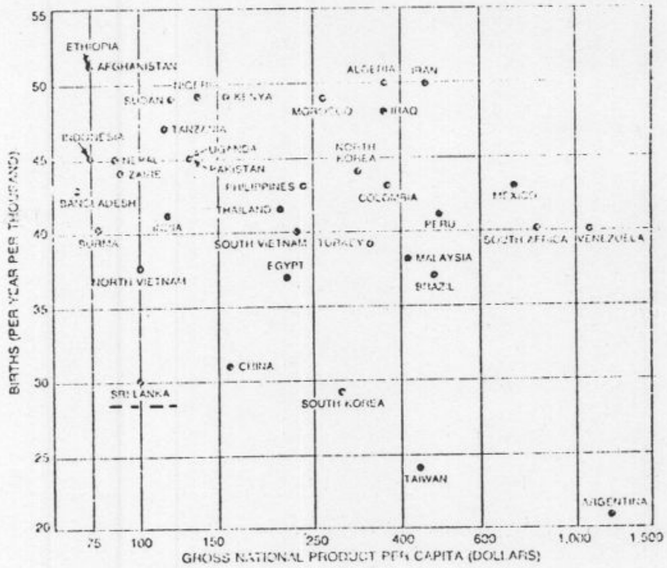

Fig. 8

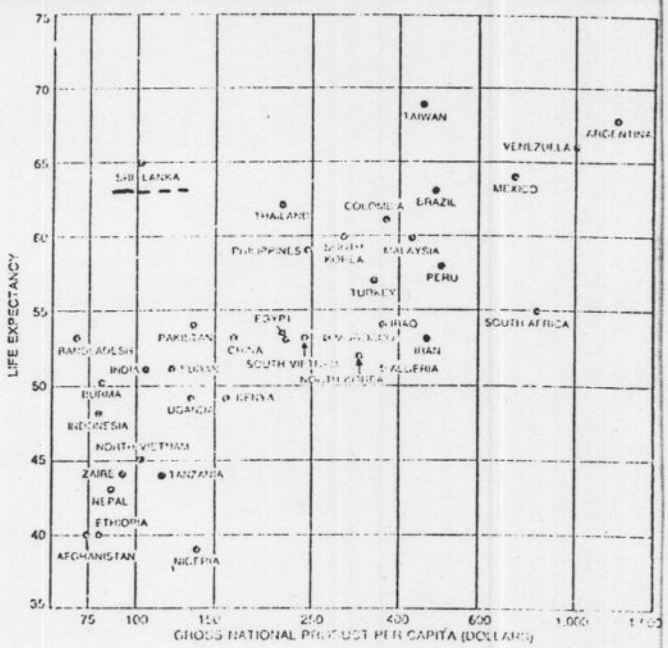

Fig. 9

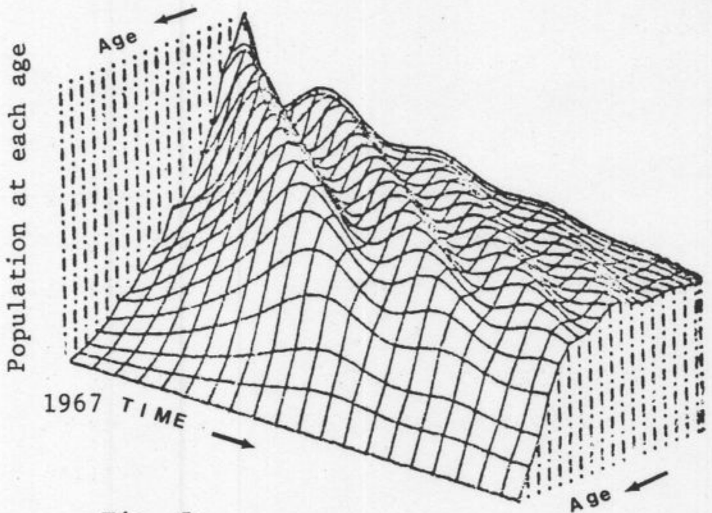

Fig. 5a

1967+150 years

Formation of population waves by rapid fall in total fertility as given in Fig. 6

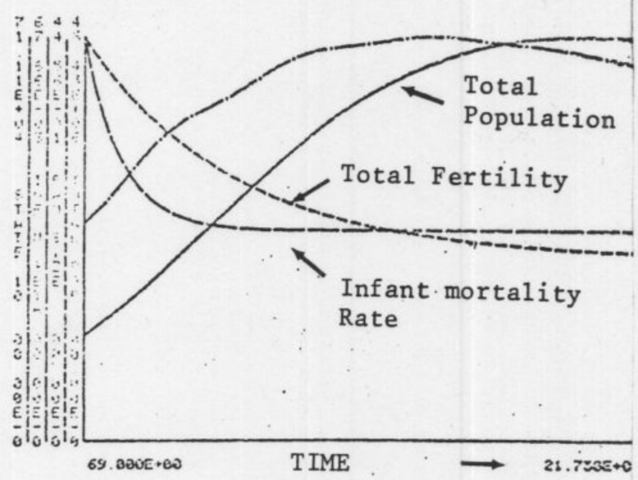

Fig. 6

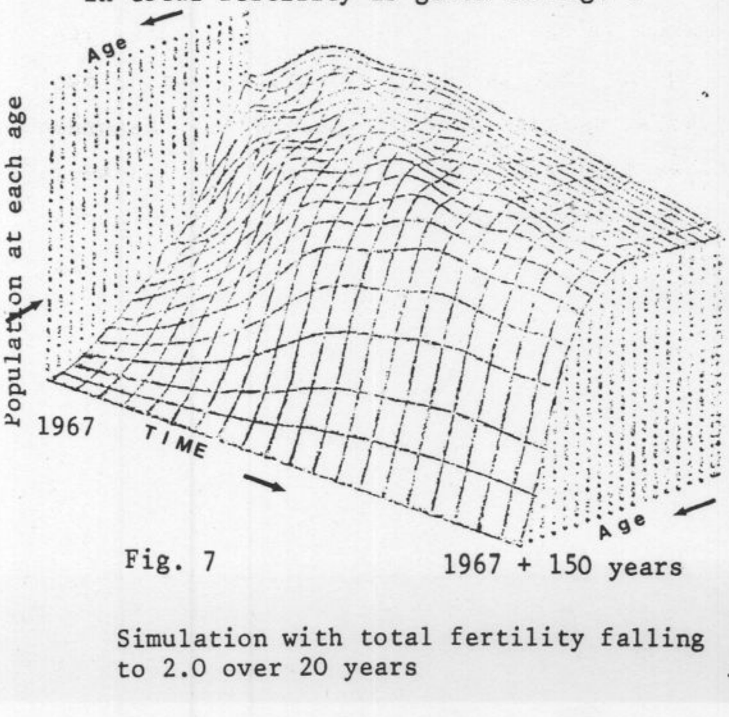

Fig. 7

1967 + 150 years

Simulation with total fertility falling to 2.0 over 20 years

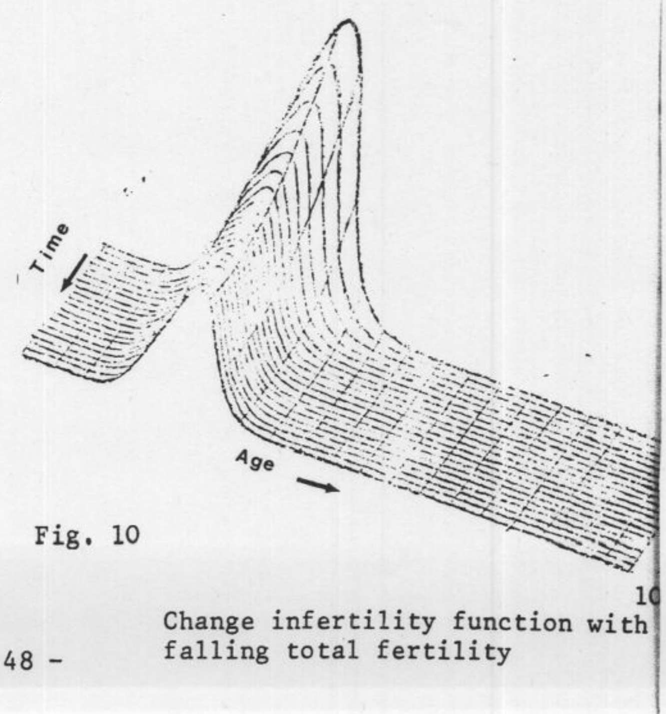

Fig. 10

Change infertility function with falling total fertility

The Simulation results shown were produced on a PDP-10 computer with a Tetronix graphic display using an experimental extended version of the 'DYSMAP' Simulation package.

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