STRUCTURAL EQUIVALENCE BETWEEN CONTROL SYSTEMS THEORY AND SYSTEM DYNAMICS

Part I

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ABSTRACT

This paper discusses an algebraic and a diagrammatic method to highlight the structural equivalence between Control Systems Theory and System Dynamics. An analogue scheme of visual representation to SD Models is suggested which makes it possible to express the SD model in the form of a vector-matrix state differential equation. An obvious duality of this representation with the signal flow graph helps computing the elements of the system matrix.

1. Introduction

It is well known that the concept of feedback is intrinsic to SD and that one of the major factors responsible for the development of SD is Feedback Control Theory (Forrester (1)). In a recent article Sharp (2) has pointed out many potential applications of Control Systems Theory (CST) in SD. Some such applications are already reported by Rademaker (3) on the World Dynamic Studies. However, mostly due to its mathematical sophistication, control systems theory has not been widely accepted by the system dynamicists.

The purpose of this and the attendant paper (4) is to highlight the structural equivalence between CST and SD. The papers are directed towards System dynamicists with limited or no exposure to CST with a hope that these will sufficiently motivate them to make use of available techniques and results of CST in the context of SD.

2. A Fundamental Difference in Modelling Approach:

In SD, the most important variable is the level variable which is created in flows and which is modelled as the initial level value plus the net flow into the level. For example, inventory (INV) created between production rate (PR) and shipment rate (SR) is modelled as

L INV.K = INV.J + (DT) (PR.JK – SR.JK)
N INV = INVI

Now if the solution time, DT, is very small (i.e. if DT tends to zero, written $DT \longrightarrow 0$), these equations may be written as an integral equation of the following form:

$$INV(t) = INV(t{=}O) + \int_o^t [PR(t) - SR(t)]\, dt$$

or,

$$INV(t) = INVI + \int_o^t [PR(t) - SR(t)]\, dt \quad \ldots\ldots(1)$$

Thus Levels are integrations and the SD modelling approach conforms to the exact direction of causation as in the real world. That means the modelling approach considers net flow [PR(t) – SR(t)] as the cause and INV as the effect as is really the fact.

If Eqn. (1) is differentiated w.r.t. time, the following is obtained.

$$\frac{d}{dt} INV(t) = PR(t) - SR(t) \quad \ldots\ldots(2)$$

Equation (2) represents the instantaneous change in INV(t).

Of course, the initial inventory is given by

$$INV(O) = INVI \quad \ldots\ldots(3)$$

Thus Eqn. (1) is structurally equivalent to Eqn. (2) and (3). CST favours Eqn. (2) and (3) rather than Eqn. (1). But in Eqn. (2) INV has been presented as the cause and [PR(t) – SR(t)] the effect. Thus CST demands mathematical formulation of systems based upon differential equations (and not integral equations as in SD) and, thereby, reverses the real direction of causation. This has earlier been pointed out by Forrester (5).

An important observation to make is that mathematics literature is replete with solution methods to differential equations (rather than to integral equations) and, hence, CST enjoys a definite advantage over SD in model analysis and design. Therefore, differential equation representation of SD models is desirable to bring about a structural equivalence between the two techniques and consequently to be able to apply some results of CST to SD.

3. Differential Equation Representation of SD Modules:

It is well known that auxiliary variables are subdivisions of rate variables. That means it is feasible to write SD model equations in terms of levels, rates and constants only. It is also known that rates are expressed in terms of levels and constants. So if one transforms the level (integral) equations into differential equations and replace rates in these equations by appropriate levels and constants, then the resultant set of differential equations completely represent the SD model. Such form of representation is the starting point for any control theoretic analysis.

The steps involved in such a representation may be summarized below:

STEP I : Express rates in terms of levels and constants only. If any auxiliary variable is smoothed, express the auxiliary variables in terms of levels and constants also.

STEP II : Replace rates in level equations by relationships containing appropriate levels and constants.

STEP III : Convert the level equations to differential equations and use the subscript, t, instead of J, K, JK, and KL.

Level variables are called *State variables* in CST, and the differential equations are called *State differential equations.* It may be pointed out here that in addition to the state differential equations, one has to specify the *initial value equations* to completely represent the model. A method to avoid explicit use of initial value equations is to substitute the level variables by the discrepancy of the level variables from their initial values.

It may be noted here that this form of representation gives no information regarding rate, auxiliary or any supplementary variables where as, in SD, many such variables may be of interest. These variables of interest may also be expressed in terms of levels and constants. Such equations are called *output equations.* Thus output equations are not necessary to generate dynamic behaviour of the system but give information on variables other than the level variables.

3.1 A Single-Order System :

Consider a singel-order SD model consisting of a negative feedback loop the influence diagram of which is given in Fig. 1.

The Dysmap equations of such a system are given by the following:

L LEV.K = LEV.J + DT * RT.JK

N LEV = 100

R RT.KL = DISC.K/TMD

C TMD = 4

A DISC.K = DLEV – LEV.K

C DLEV = 1000

One may follow the steps outlined earlier to obtain the state differential equation.

STEP I : RT.KL = (DLEV – LEV.K)/TMD

STEP II : LEV.K = LEV.J + DT *(DLEV – LEV.J)/TMD

STEP III : $$\frac{d}{dt}LEV(t) = -\left(\frac{1}{TMD}\right) LEV(t) + \frac{DLEV}{TMD} \qquad \ldots\ldots(3)$$

The initial value equation is of course given by

$$LEV(o) = 100 \qquad \ldots\ldots(4)$$

Eqn. (3) and (4) completely represent the single-order system.

$$\text{Defining } lev(t) \triangleq LEV(t) - LEV(o) \qquad \ldots\ldots(5)$$

and replacing LEV(t) by lev(t) +LEV(o), Eqn. (3) may be written as the following:

$$\frac{d}{dt}lev(t) = -\frac{1}{TMD}lev(t) + \frac{DLEV - LEV(o)}{TMD} \qquad \ldots\ldots(6)$$

Thus Eqn. (6) may be written as a substitute for Eqn. (3) and (4).

A passing remark may be made about Eqn. (3) and (6). A state equation in terms of the transformed state variable may bring in some more terms in the R.H.S. Also if the initial value of a transformed state variable is zero, it need not be specified. For example initial value of lev(t) = lev(o) = LEV(o) – LEV(o) = O, so it need not be written explicitly.

3.2 A First-Order Delay:

A first-order exponential delay is represented by the following SD equations:

L LEV.K = LEV.J + DT *(IN.JK – OUT.JK)

N LEV = (IN) (DEL)

R OUT.KL = LEV.K/DEL

C DEL = 6

Following the three steps outlined earlier the equivalent state differential equation is given by

$$\frac{d}{dt}Lev(t) = IN(t) - \frac{1}{DEL}LEV(t) \qquad \ldots\ldots(7)$$

The initial value equation is of course

$$LEV(o) = IN(o)*DEL \qquad \ldots\ldots(8)$$

Usually, the outflow of a delay, and not the level stored in a delay, is used elsewhere in the system. So a differential equation in terms of the outflow seems desirable.

Defining OUT(t) = LEV(t)/DEL, Eqn. (7) and (8) may be written as

$$\frac{d}{dt}OUT(t) = \frac{1}{DEL}IN(t) - \frac{1}{DEL}OUT(t) \qquad \ldots\ldots(9)$$

and

$$OUT(o) = IN(o) \qquad \ldots\ldots(10)$$

Again defining out(t) = OUT(t) – OUT(o), Eqn. (9) and (10) may be written as the following:

$$\frac{d}{dt}out(t) = \frac{-1}{DEL}out(t) + \frac{1}{DEL}[IN(t) + OUT(o)] \qquad \ldots\ldots(11)$$

Since the initial value of out(t) = OUT(t=o) – out(o) = O, the initial value for out(t) is not necessary to be explicitly written.

Eqn. (9) and (10) may be rearranged in the following way:

$$\frac{d\,out(t)}{dt} = \frac{1}{DEL}\,[IN(t) - OUT(t)] \qquad \ldots\ldots(12)$$

$$out(o) = IN(o) \qquad \ldots\ldots(13)$$

Eqn. (12) and (13) are equivalent to SMOOTH equations in SD. This tallies with our previous knowledge that SMOOTH equations are equivalent to first-order exponential delays.

3.3 A Third-Order Exponential Delay

A third-order exponential delay is equivalent to three first-order exponential delays with each constant, $\frac{DEL.}{3}$

Following the three steps, the state differential equations for a third-order delay may be written as the following:

$$\left.\begin{aligned} \frac{d}{dt}L1(t) &= IN(t) - \frac{3}{DEL}L1(t) \\ \frac{d}{dt}L2(t) &= \frac{3}{DEL}L1(t) - \frac{3}{DEL}L2(t) \\ \frac{d}{dt}L3(t) &= \frac{3}{DEL}L2(t) - \frac{3}{DEL}L3(t) \end{aligned}\right] \qquad \ldots\ldots(14)$$

Initial value equations are of course given by

$$L1(o) = L2(o) = L3(o) = IN(o) * (DEL/3) \qquad \ldots\ldots(15)$$

L1, L2 and L3 are the three levels in the third-order delay.

The number of differential equations will always equal the order of the system. The three equations in Eqn. (14) may be represented in a vector-matrix form as the following:

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{b}\,z\,(t) \qquad \ldots\ldots(16)$$

where,

$$\underline{x}(t) = \begin{bmatrix} L1(t) \\ L2(t) \\ L3(t) \end{bmatrix}; \quad A = \begin{bmatrix} \frac{-3}{DEL} & 0 & 0 \\ \frac{3}{DEL} & \frac{-3}{DEL} & 0 \\ 0 & \frac{3}{DEL} & \frac{-3}{DEL} \end{bmatrix}; \quad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\underline{x}}(t) = \frac{d}{dt}\begin{bmatrix} L1(t) \\ L2(t) \\ L3(t) \end{bmatrix} = \frac{d}{dt}\underline{x}(t); \quad z\,(t) = IN(t).$$

Eqn. (16) is called the vector-matrix state differential equation and is a popular way of representing systems in CST.

Since the outflow rate from a delay is used elsewhere in a system it may be desirable to define rates to be state variables. Hence, one can proceed as in §3.2 to get the transformed state vector-matrix state differential equation.

4. Vector–Matrix State Differential Equation of an SD Model:

An example from Coyle (6) is used here for the sake of illustration. The influence diagram for the base run of the example is reproduced in Fig. 2.

It may be observed in Fig. 2 that TRBL is a table function which is actually nonlinear in shape. The actual DYSMAP equations used for the base run are the following:

```
A RBL.K = TABHL (TRBL,AOR.K,     ]
                 50, 150, 25)    ]
                                 ]   ......(17)
T TRBL = 400/525/600/650/675     ]
```

For the sake of simplification the following equation may be written for RBL:

```
A RBL.K = (AOR.K) (TMRBL)          ]
D TMRBL = (WK) Time for            ]
               Required Backlog    ]   ......(18)
C TMRBL = 6                        ]
```

Elaborate methods to transform nonlinearity to linearity are studied in the associated paper (4).

Following the three steps indicated earlier the vector-matrix state differential equation of this model may be written as

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{b}z(t) \qquad \ldots\ldots (19)$$

where,

$$\underline{x} = \begin{bmatrix} INV \\ ASR \\ AOR \\ OBL \\ APL \\ L1 \\ L2 \\ L3 \end{bmatrix}, B = \begin{bmatrix} -1 \\ \frac{1}{TASR} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and } z(t) = SR(t)$$

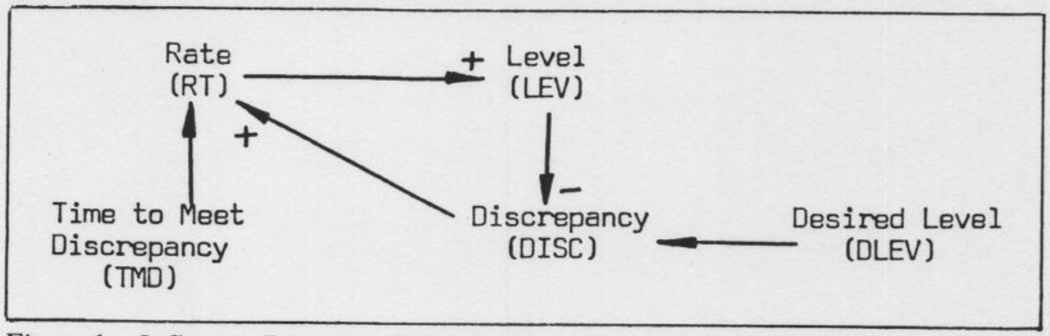


Figure 1 Influence Diagram of a Single-Order Negative System

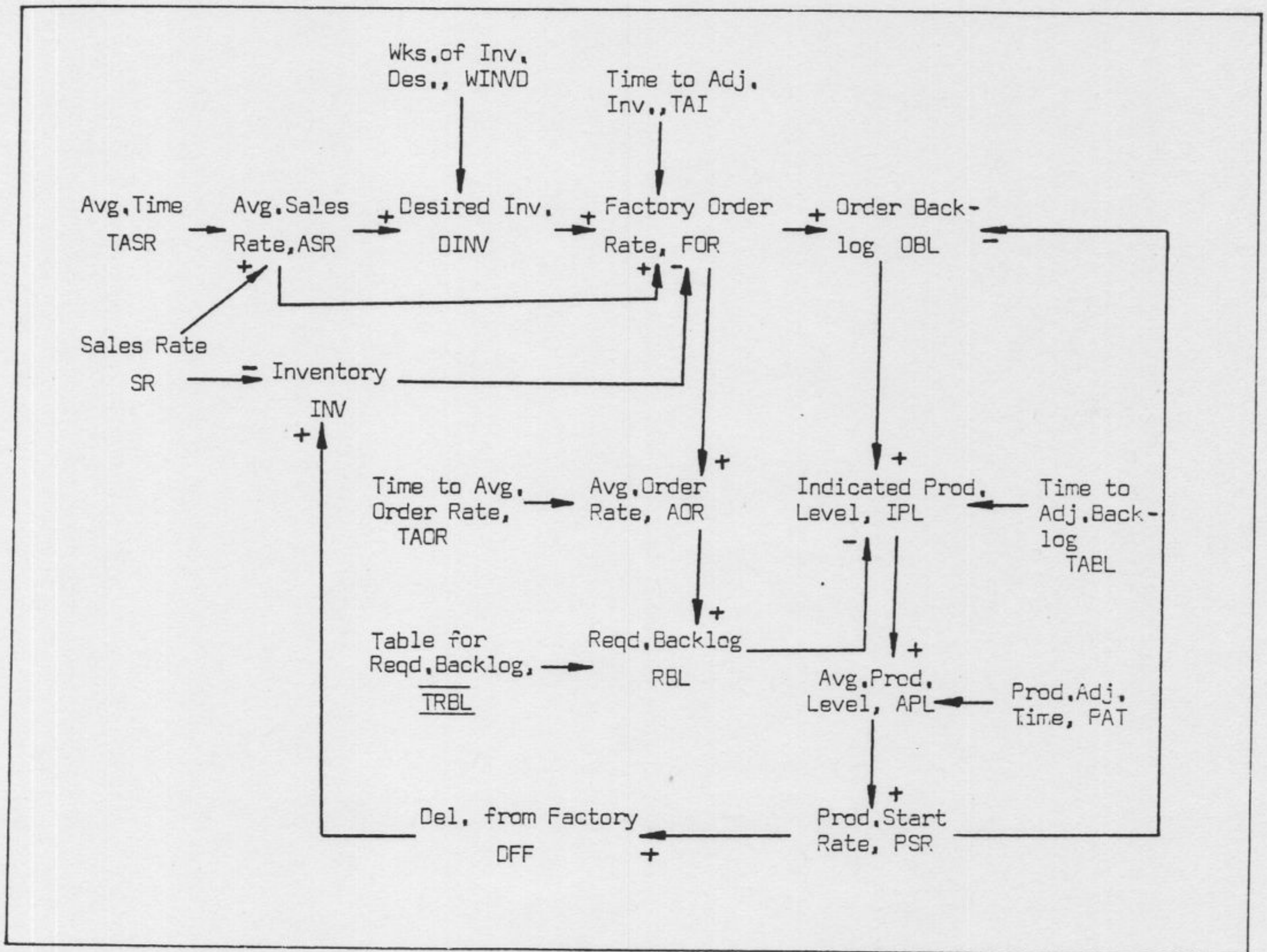


Figure 2 Influence Diagram of the Example under Consideration

$$
A=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{PDEL} \\
0 & \frac{-1}{TASR} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{(TAOR)(TAI)} & \frac{TAI+WINVD}{(TAOR)(TAI)} & \frac{-1}{TAOR} & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{TAI} & \frac{TAI+WINVD}{TAI} & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & \frac{-TMRBL}{(TABL)(PAT)} & \frac{1}{(PAT)(TABL)} & \frac{-1}{PAT} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{-3}{PDEL} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{PDEL} & \frac{-3}{PDEL} & 0 \\
0 & 0 & 0 & 0 & 0 & & \frac{3}{PDEL} & \frac{-3}{PDEL}
\end{bmatrix}
$$

It may be indicated here that considerable effort is called for representing an SD model in the form of state differential equations. Eqn. (19) can generate dynamic behaviour of interest. But it gives no information on auxiliary or rate variables of interest. One has, therefore, to use the output equations. Thus if Factory Order Rate (FOR) and Production Start Rate (PSR) are the non-state variables of interest, the corresponding output equations are given by the following:

$$PSR(t) = APL(t)$$

$$FOR(t) = \frac{TAI+WINVD}{TAI} *ASR(t) - \frac{1}{TAI} *INV(t) \qquad \ldots\ldots (20)$$

In the vector-matrix form this may be written as

$$\underline{y}(t) = C\underline{x}(T) \qquad \ldots\ldots (21)$$

where,

$$\underline{y} = \begin{bmatrix} PSR \\ FOR \end{bmatrix}, \quad C = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ \frac{-1}{TAI}, & \frac{TAI+WINVD}{TAI}, & 0, & 0, & 0, & 0, & 0, & 0 \end{bmatrix}$$

5. **General Forms of State Differential Equation and Output Equations:**

In Eqn. (19), there exists only one exogenous variable, z(t). But in general, there may be $q, q \geq o$, number of exogenous variables. Also there may be some constant vectors as present in Eqn. (3) and (6). Hence, the general form of a state differential equation is the following:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{z}(t) + \underline{e} \qquad \ldots\ldots (22)$$

and the corresponding general form of an output equation is

$$\underline{y}(t) = C\underline{x}(t) + D\underline{z}(t) + \underline{f} \qquad \ldots\ldots (23)$$

where,

$\underline{x}(t)$ = (nx1) state variable vector

A = (nxn) system matrix

B = (nxp) disturbance matrix

$\underline{z}(t)$ = (px1) exogenous variable vector

$\underline{e}$ = (nx1) constant vector

$\underline{y}(t)$ = (qx1) output variable vector

C = (qxn) output matrix

D = (qxp) matrix

$\underline{f}$ = (qx1) constant term vector

Of these vectors and matrices the A-matrix is the most important because it is responsible for generating the dynamic behaviour which is internal to the system. But unfortunately, the computation of the terms in the A-matrix requires considerable time and effort. An analogue scheme of visual representation of SD models is given below which helps in easy computation of the A-matrix.

6. **An Analogue Scheme of Visual Representation of SD Models:**

Causal loop diagrams, flow diagrams and influence diagrams are various popular methods of visual representation of SD models. Some criticisms levied against these representations may be summarized below:

a) These diagrams do not portray the precise mathematical relationships

b) These diagrams do not show the initial values of the levels

c) It is difficult to identify the feedback loops due to the presence of many lines intersecting one another.

Based on the analogue representation scheme in Control Systems Theory a new diagram may be used with the help of a few symbols. Table 1 gives the symbols and their implications. These symbols are a modified and extended version of symbols originally proposed by Rademaker (3).

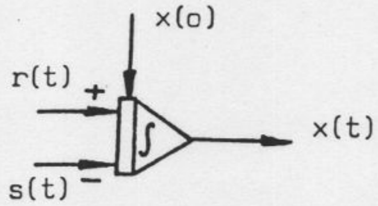
The single-order SD model of § 3.1 is diagrammed and shown in Fig. 3.

The analogue representation of a first order delay of § 3.2 is shown in Fig. 4.

Since third order delays are extensively used in practice and since the outflow rate depends solely on the inside level stored

TABLE 1

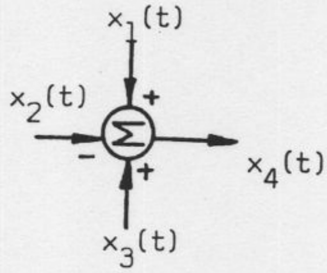
Implications



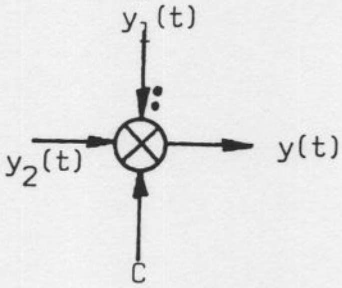
$$x(t) = x(o) + \int_o^t \left[r(t) - s(t) \right] \; dt$$

$\equiv \dot{x}(t) = r(t) - s(t)$

x(o) is also given

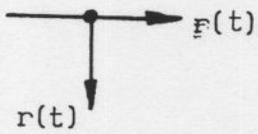


$$x_4(t) = x_1(t) - x_2(t) + x_3(t)$$

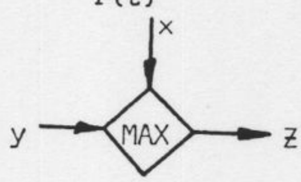


$$y(t) = c^* y_2(t)/y_1(t)$$

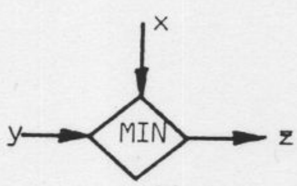
y(t) is a TABHL/TABLE function of x(t)



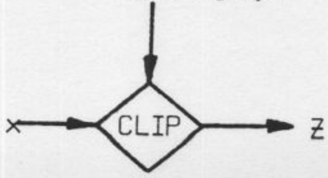
Information is taken off the main stream, r(t)



Z = MAX (x,y)



Z = MIN (x,y)



Z = CLIP(x,y,y,x)

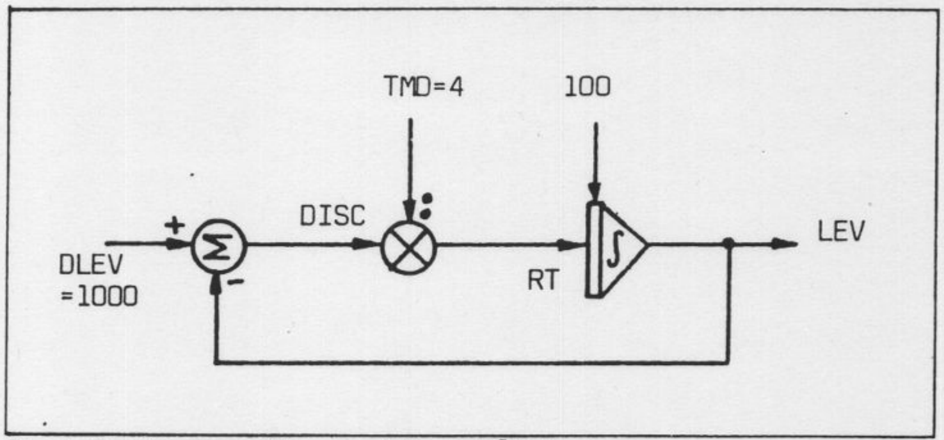


Figure 3: Analogue Representation of § 3.1

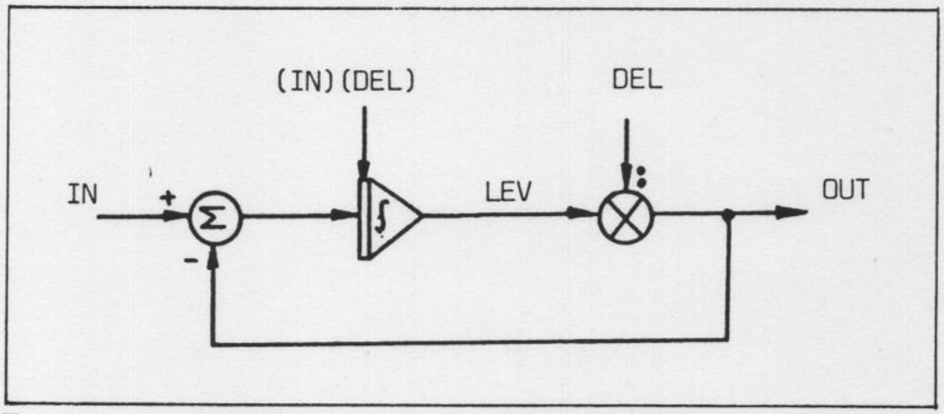


Figure 4: Analogue Representation of First-Order Delay

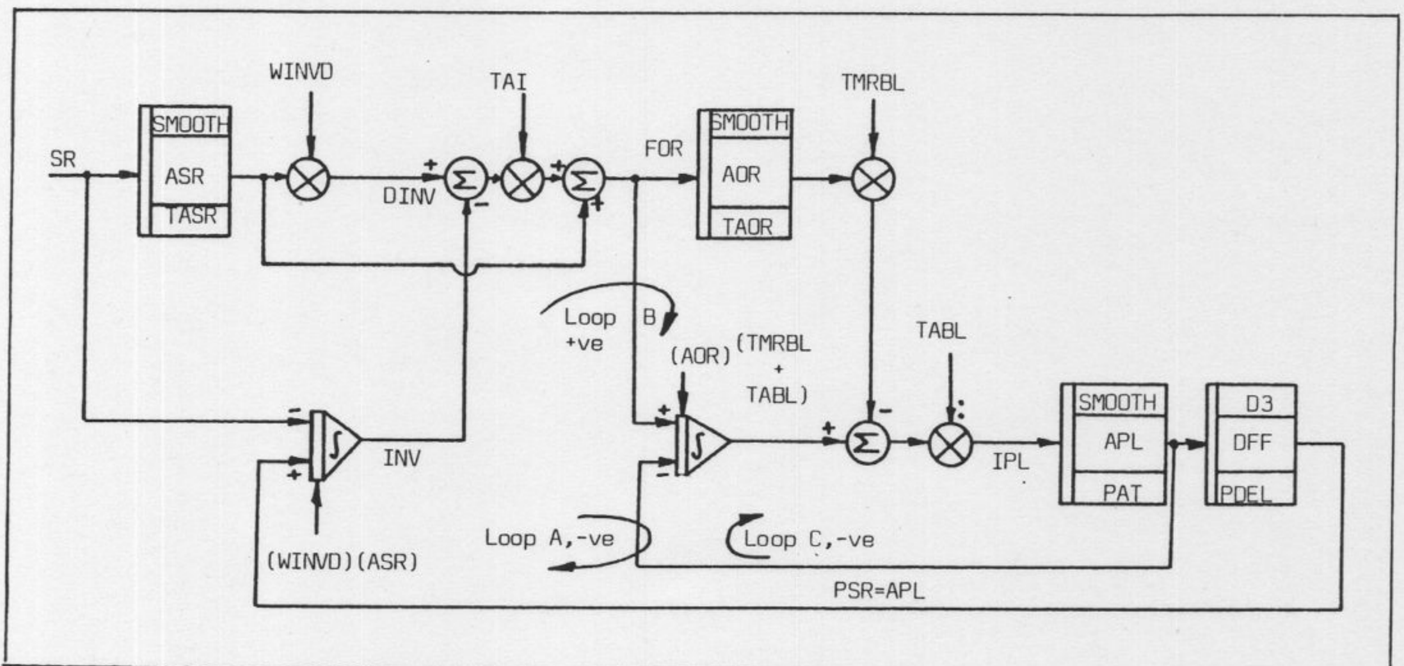


Figure 5: Analogue Representation of § 4

in the delay, a short hand symbol may be retained for the purpose. In fact Forrester's original symbol may be retained. For similar reasons Forrester's original symbol for DELAYn, SMOOTH, DLIFNn may also be retained.

The analogue representation of § 4 along the lines of the suggested scheme is given in Fig. 5.

It may be observed from Fig. 5 that the flow proceeds from right to left whenever a feedback occurs. Therefore, all the loops are very distinct. The three loops in the model are very obvious. It is also very easy to get various information such as gains, and delays etc. in each loop as obtained by Coyle (6) (page 211). It also overcomes the criticisms against the popular visual representation of SD models.

It may be pointed out at this stage that the symbols represent operations and arrows represent variables. The constants associated with a ⊗ – symbols define gains between two variables associated with corresponding ⊗ – symbols. For example gain between ASR and DINV equals WINVD and gain between AOR and IPL equals (TMRBL/TABL), the negative sign appearing here because of the presence of such a sign in the sequence of arrows from AOR to IPL. Gain across a smooth-operation is not obvious from the diagram. But it is known that a smoothed level equation is equivalent to a first-order delay represented by Eqn. (11) from which it is obvious that the path gain across the smoothing operation is the reciprocal of the smoothing time constant. For the same reason, the path gain between inflow and outflow rates for a first-order exponential delay also equals the reciprocal of the delay constant. Similar statements cannot of course be made about third-order delays since there exists three hidden state variables.

It was earlier stated that it would be possible to compute the entries in the A-matrix directly from such an analogue scheme. For that purpose, the following definitions are made considering a direct analogy between this form of representation and the signal flow graphs.

A *Path* between two state variables is a series of arrows from head-to-tail without passing through any state variable.

Path gain (or transmittance) is the multiplication of all gains along the path. Whenever negative signs appear on the path, this must be considered as a gain of -1.

Equivalent gain (or transmittance) between two state variables is given by sum of gains of all parallel paths existing between the two state variables. This equals the corresponding entry in the A-matrix.

Consider the transmittance between ASR and AOR. There are two paths existing between the variables ASR and AOR.

$$\text{Path gain of Path 1} = \frac{WINVD}{TAI} * \frac{1}{TAOR}$$

$$\text{Path gain of Path 2} = \frac{1}{TAOR}$$

so equivalent gain between ASR and AOR

$$= \frac{WINVD}{TAI} * \frac{1}{TAOR} + \frac{1}{TAOR} = \left(\frac{WINVD}{TAI} + 1\right) * \frac{1}{TAOR}$$

which, of course, appears on the corresponding entry in the A-matrix.

One may proceed in the similar fashion and verify that the equivalent path gains between two state variables are in fact the corresponding entries in the A-matrix. If there exists no path between two state variables then the path gain equals zero.

7. Conclusion

It is shown in this paper that an SD model with linear relationships can be transformed to the form of a vector-matrix state differential equation, which forms the starting point in a control theoretic analysis. The procedures to be followed in the presence of nonlinearities are discussed in the accompanying paper (4).

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