COMPLEX BEHAVIOUR IN SYSTEM DYNAMICS MODELS

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ABSTRACT
This paper shows how irregular, more-or-less random fluctuations arise in deterministic economic models. The heart of the matter lies in the intrinsic properties of nonlinear feedback, a phenomenon emphasized by Forrester as causing fundamental difficulties in conventional dynamic analysis.

1. OBSERVED IRREGULARITY

In spite of progress in theory, econometrics, and computer simulation, the prospects for accurate prediction of significant economic variables – beyond a few months in advance – seem as remote now as ever. Exasperating irregularity is exhibited by familar monetary variables often charted on the financial pages such as the prime rate, the money supply, and the Commerce Department's index of 500 stock prices. Although the last of these is widely regarded as exhibiting properties of a random walk, the first two are generally thought to be determined by monetary policy and systematic economic forces. Obviously, however, changes in money supply are erratic and, while movements in the prime rate could be closely approximated by a smooth curve, the recent history is irregular and characterized by cusp-like reversals at uneven intervals. "Real" variables such as unemployment, industrial output, aggregate capacity, and business inventories likewise show an unhappy tendency toward wandering fluctuations.

It has been customary to attribute such behavior to random, exogenous "disturbances" or "shocks" that continually perturb system behavior from a regular path with the result that system states are erratic and more or less unpredictable. An alternative explanation is that unpredictable irregularity is generated endogenously, without any intervention from exogenously determined stochastic elements, by the interplay of technology, preferences and behavioral rules alone. The latter view is surely the more satisfactory one on scientific grounds for it is precisely the task of theory to show how the character of observed events is intrinsic to the process generating phenomena. It is this endogenously generated, intrinsic irregularity that is the subject of this paper.

2. SIMULATED IRREGULARITY

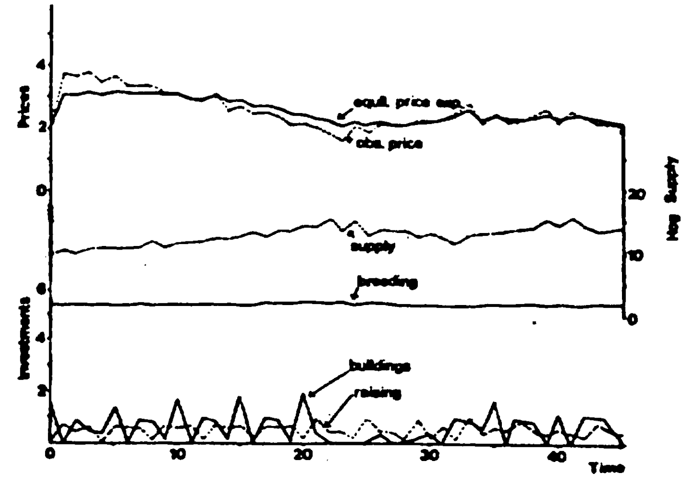
2.1 The Generalized Cobweb Model

Irregularities of the type we are talking about began to arise in computer simulation studies of deterministic recursive programming models in the early sixties, nearly two decades ago. Those models were developed for the purpose of explaining and projecting behavior of industrial sectors, agricultural regions, or individual firms (Day [1963], Day and Cigno [1978]).

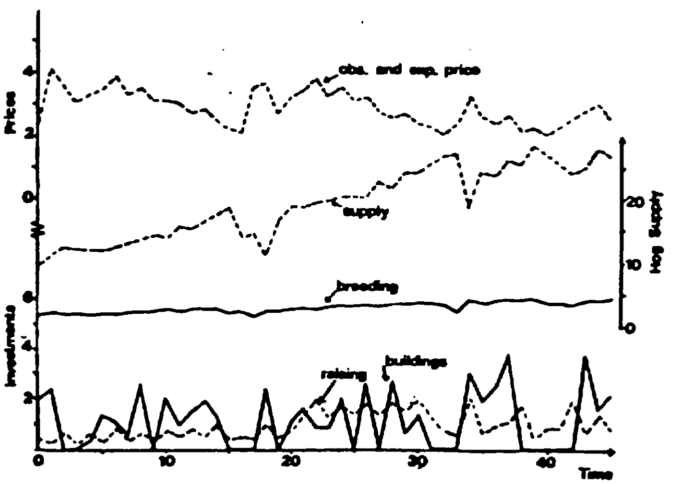
An example chosen here for illustrative purposes is the Generalized Cobweb Model described by Muller and Day (1978). The central assumptions of this model are (1) producers maximize an anticipated, discounted profit stream over a multi-period horizon subject to various constraints; (2) the constraints include the usual technical restrictions on the use of land, labor, and physical capital, "adaptive" behavioral constraints representing "local search" and caution in response to current opportunities and a financial constraint limiting financial capital to "reinvestment income" or savings and externally constrained borrowing; (3) producers anticipate prices according to an adaptive expectations forecast, and (4) realized "market" prices are generated by a deterministic demand function. The complete model represents behavior by a rolling-plan with market feedback. It was designed to simulate farm production and investment behavior and market price and sales behavior for typical agricultural commodities.

In order to narrow the scope of the investigation the parameters were chosen to reflect cost and demand conditions for hog production in West Germany. Deterministic runs of the model were obtained for various conditions of demand (e.g. with and without a trend component) and for varying estimates of the coefficients pertaining to behavior under uncertainty. In Figure 1 the results of two such runs are reproduced. Of particular interest is the trajectory of investment in buildings, which shows a pronounced fluctuation with no tendency of converging to an oscillation of fixed period. Indeed, the apparently quite stable pattern in Figure 1a that is repeated three times between period 5 and period 20, suggesting a five-period cycle, is broken thereafter. Through the fiftieth iteration no similar regularity has yet appeared. In Figure 1b, where an exponential demand shifter has been introduced, the amplitude and the irregularity of the fluctuations seem more pronounced.

These results suggest that agricultural commodities might very well exhibit quite irregular oscillations in output and prices even in the absence of more or less random shocks from weather and political events. Similar results have been obtained for closed recursive programming models of other sectors. A recent example is the "EDEM" model of a prototypical regulated electric utility. See Day (1982). There, erratic substitutions between a capital intensive and a capital saving technique for generating electric power are shown to emerge after a long period of efficacious growth.



(a) Stationary Demand



(b) Exponential Demand Shifter

Figure 1: Simulations of the Generalized Cobweb Model Muller and Day (1978, pp. 242-247)

2.2 A Reductionist Approach: The Robertson-Williams Cobweb Model

The characteristics of the Generalized Cobweb Model responsible for fluctuations in prices, output and investment are: (1) the presence of a financial constraint that depends on lagged revenues, (2) the dependence of revenue on a demand function with variable elasticity, and (3) the independence of pricing from the production and investment decisions: "market" prices are determined by a purely competitive market clearing process. After a period of growth (perhaps), output levels eventually reach the inelastic portion of the demand curve; consequently, revenues fall. This reduces working capital and borrowing ability for the subsequent period. Production and/or investment must be reduced or a shift to money-saving production and investment effected. Later, because market supplies are reduced, prices increase and revenues recover. All this happens the way it does because of the nonlinear feedback involved in the working capital equation. By stripping away all of the features of the generalized Cobweb model except the three characteristics that cause cycles, we arrive at a single equation model that proves to be quite tractable and for which precise analytical and constructive conditions for erratic behavior can be given. The basic assumption is that current expenditures come from previous income, the so-called Robertsonian Lag. In the case of the firm, a sales maximizing hypothesis (Baumo [1959]) coupled with a financial constraint that requires all production costs to be limited to previous sales revenues leads to such an equation (Day [1967], Williams [1967]).

Assuming that all firms in a purely competitive industry possess identical unit costs, total output obeys the difference equation: Present Total Cost $\equiv cX_{t+1}$ = Lagged Total Revenues $\equiv X_t D(X_t)$, or

$$(1) \quad X_{t+1} = X_t D(X_t)/c$$

in which c is the constant unit cost of production for each firm and $D(\cdot)$ is the inverse industry demand curve.

Given the simplifications we have imposed, the dynamics of industry output must depend on unit cost and on the parameters of demand. This is most easily seen by considering the special case in which $D(X) \equiv a - bX$ so that (1) becomes

$$(2) \quad X_{t+1} = X_t (a - bX_t)/c.$$

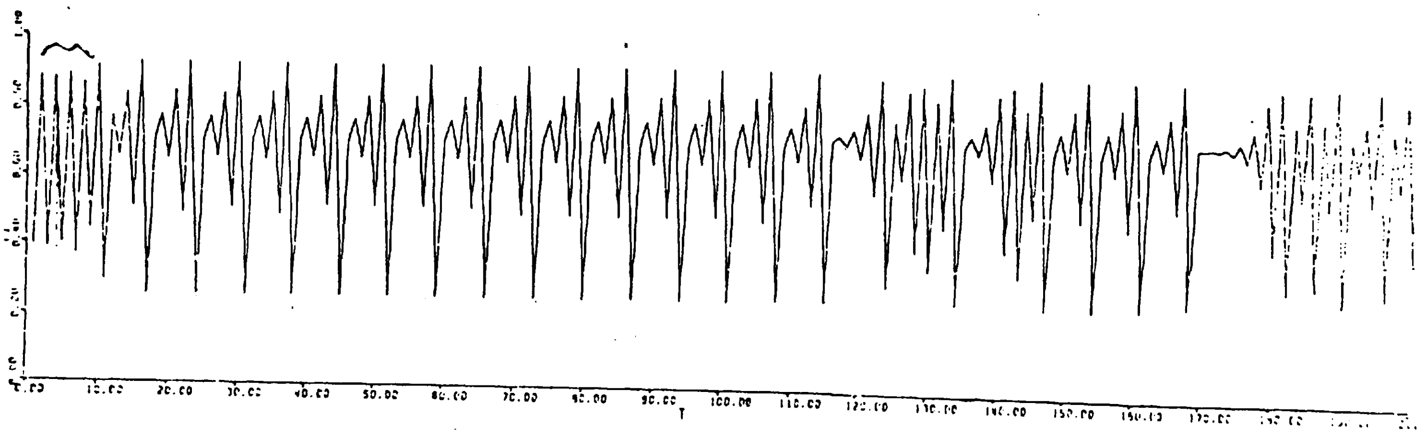
Setting $x = bX/a$ we obtain the difference equation

$$(3) \quad x_{t+1} = mx_t(1 - x_t),$$

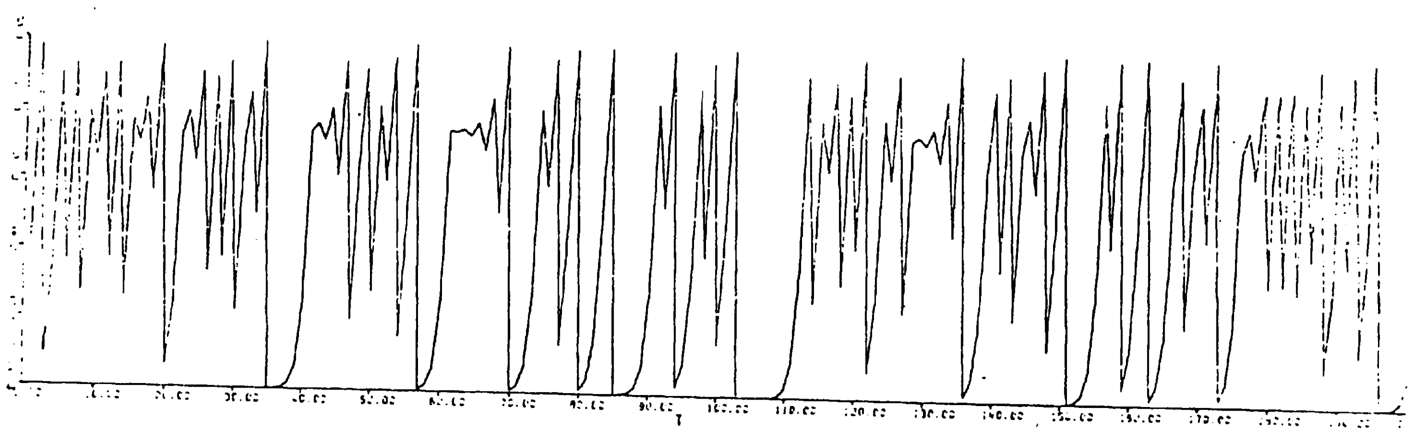
where m = a/c. The parameter "m" has a simple intuitive meaning: it is the extent of the market "a" divided by an "efficiency index" we could say that "m" is the "extent of the market" measured in "cost efficiency" units. Elsewhere I have shown how industry behavior becomes increasingly complex as m increases (Day [1982b]). Just how complex is shown in Figure 2. Diagram (a) shows a two-period oscillation that appears to jump to a seven-period cycle. The latter reproduces itself fifteen times only to wander away in a quite irregular pattern. Diagram (b) develops extreme fluctuations with periods of growth followed by cycles and occasional plunges at erratic intervals that nearly eliminate the industry altogether.

As one searches through these graphs, instances can be found in which the successors of two visually identical but non-adjacent points, such as those positive values very close to zero, depart rapidly in value. To put it another way the histories following two visually identical points are not identical!

At the time (circa 1963-1965) it seemed to me *that such* irregularities would give way eventually to at least quasi-periodic motion. (Of course, on a digital computer with finite memory periodic cycles *must* occur but that is an artifact of computer design). The thought that histories might not repeat



(a) Chaos (a/c = 3.7037)



(b) Chaos (a/c = 4.0)

Figure 2: Simulations of the Robertson-Williams Cobweb Model

any fixed pattern – even approximately – did not occur to me then.

3. THE WORK OF LORENZ: NON-PERIODIC FLOW, INSTABILITY AND STRANGE ATTRACTORS

In an independent line of work, however, exactly that possibility was discovered and investigated in considerable depth. Indeed, equation (3) had been arrived at already by a quite different route. Edward Lorenz, in a brilliant series of papers (Lorenz [1963a], [1963b], [1964a], [1964b]), developed a theory of "deterministic, nonperiodic flow" on the basis of differential equations designed to represent the properties of forced, dissipative, hydrodynamical systems, and which was motivated by the desire to understand salient features of the weather, in particular, its apparent property of never repeating its past history exactly. A set of differential equations was first investigated. Next, Saltzman's 3rd order special case (the "equations of convection") were examined for specific parameter values. They were shown to exhibit the phenomen of interest. Then, in a still bolder reduction, Lorenz (1964) found in equation (3) the simplest possible analog of the general "forced dissipative system" that produced unstable, erratic fluctuations of the kind shown in Figure 2.

3.1 Forced, dissipative flow

Lorenz began with a brief description of a reductionist modelling philosophy.

> A closed hydrodynamical system of finite mass may ostensibly be treated mathematically as a finite collection of molecules – usually a very large finite collection – in which case the governing laws are expressible as a finite set of ordinary differential equations. These equations are generally highly intractable, and the set of molecules is usually approximated by a continuous distribution of mass. The governing laws are then expressed as a set of partial differential equations, containing such quantities as velocity, density, and pressure as dependent variables.

It is sometimes possible to obtain particular solutions of these equations analytically, especially when the solutions are periodic or invariant with time, and, indeed, much work has been devoted to obtaining such solutions by one scheme or another. Ordinarily, however, nonperiodic solutions cannot readily be determined except by numerical procedures. Such procedures involve replacing the continuous variables by a new finite set of functions of time, which may perhaps be the values of the continuous variables at a chosen grid of points, or the coefficients in the expansions of these variables in series of orthogonal functions. The governing laws then become a finite set of ordinary differential equations again, although a far simpler set than the one which governs individual molecular motions. Lorenz (1963a, p.78-79).

He then went on to describe a general class of ordinary differential equations that would be appropriate for modelling hydrodynamic phenomena. These were the equations of "forced, dissipative flow" typified by the quadratic differential equations

$$\dot{x}_i = \Sigma_{jk} a_{ijk} x_j x_j - \Sigma_j b_{ij} x_j + c_i, \quad i, j, k = 1, \ldots, n, \tag{4}$$

in which the linear term represents the dissipative flow and the constant term represents the forcing action of, let us say, a constant inflow of energy or resources. The first term represents the nonlinear interaction among the state variables and in the absence of forcing and dissipation is assumed to allow for bounded orbits on an ellipsoid.

The equations of forced, dissipative systems were also given a central place in Prigogines's (1962) thermodynamics of irreversible processes and since developed into a general theory of self-organizing systems (Nicolis and Prigogine [1977]). The application of these ideas to urban-regional dynamics by Allen and Sanglier (1979) is briefly suggestive of the complex dynamics about which we are talking.

Lorenz proved a few basic results on existence and stability. More important, he emphasized that nonperiodic solutions of (4), should they exist, would have to be unstable.

The result has far-reaching consequences when the system being considered is an observable nonperiodic system whose future state we may desire to predict. It implies that two states differing by imperceptible amounts may eventually evolve into two considerably different states. If, then, there is any error whatever in observing the present state – and in any real system such errors seem inevitable – an acceptable prediction of an instantaneous state in the distant future may well be impossible. Lorenz (1963a, p.81).

3.2 Lorenz's Chaos Equations

Proving the existence of nonperiodic solutions, however, is not an easy task. In order to make progress Lorenz considered a special case of (4) derived by a series of transformations from Saltzman's equations of convection. These equations, which now usually bear Lorenz's name in the mathematical work that they precipitated, have the form,

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned} \tag{5}$$

The causal lags and dynamo flow diagrams for the Lorenz equations are shown in Figures 3 and 4 respectively. The

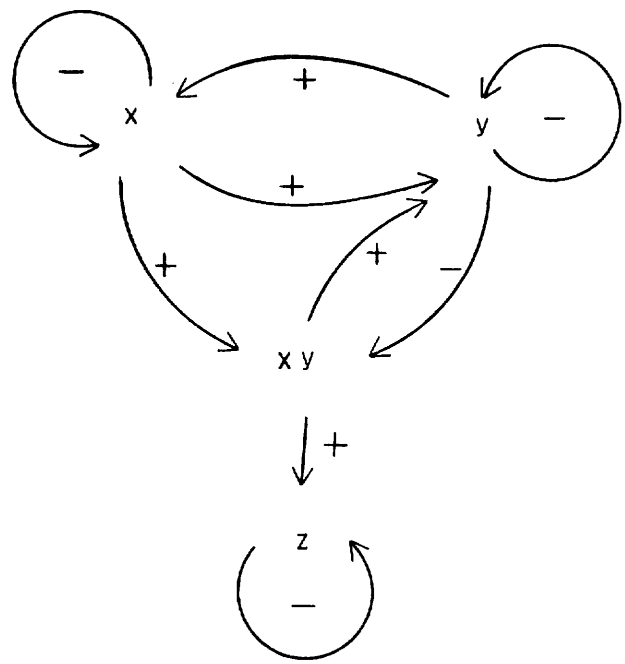


Figure 3: Causal Loop Diagram for the Lorenz Equations Nonlinearity enters through the interaction term xy which effects state variables y and z through feedback.

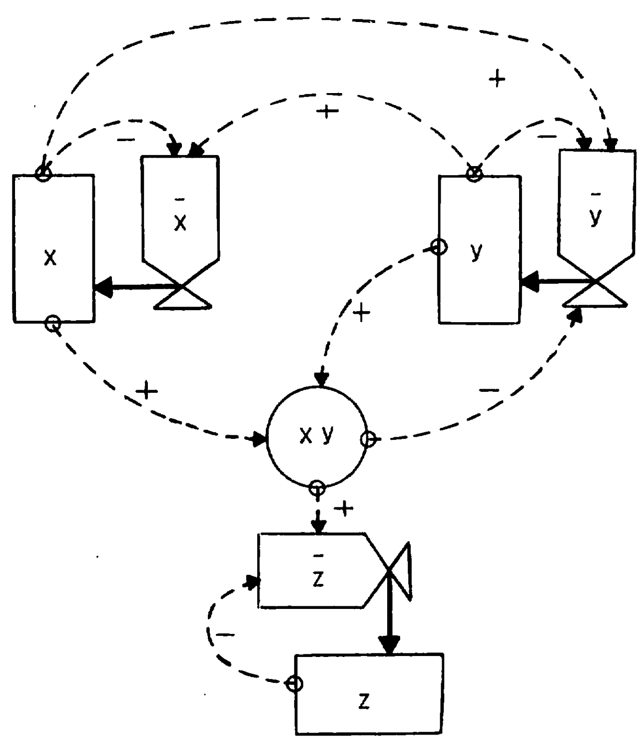


Figure 4: Dynamo Flow Diagram for the Lorenz Equations

numerical integrations of (5) for parameter values $\sigma = 10$, $r = 28$ and $b = 8/3$ led to startling results. Figure 5 for example taken from Lorenz's original paper shows the calculated trajectory for y as a function of time for the first 1000 numerical iterations of the approximating difference equations.

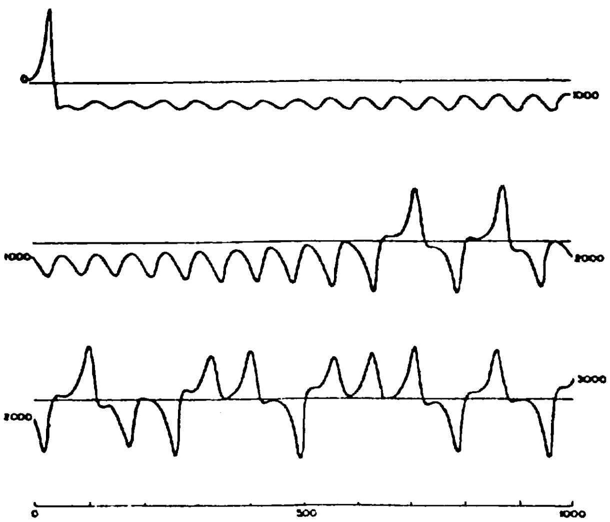


Figure 5: A Trajectory of Y for the Lorenz Equations Source: Lorenz (1964a, p.85)

Figure 6, which is reproduced from an expository article by David Ruelle (1980), is a computer drawn, two dimensional perspective of the solution from which Figure 5 was obtained. The complicated pattern of reversing spirals already well understood by Lorenz is beautifully illustrated.

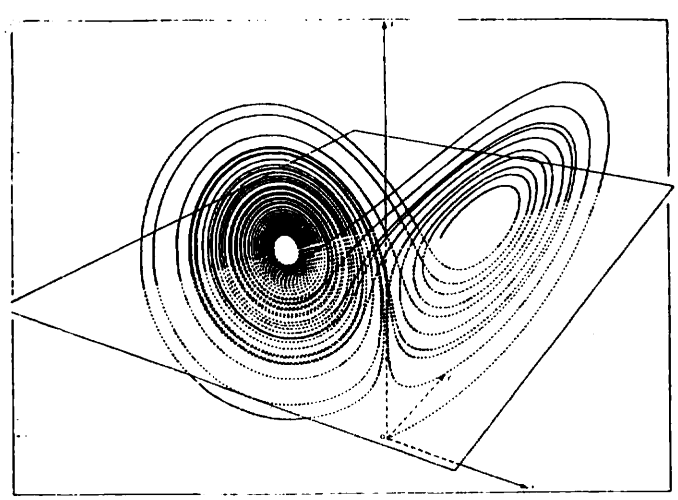


Figure 6: The Lorenz Attractor, Source: Ruelle (1980, p.133) after a computer drawing by Oscar Lanford

3.3 Rossler's Chaos Equation

A considerable amount of analytical and numerical work has since been performed on Lorenz's equations or on close relatives of them. Rossler (1976), for example, considered the set of equations

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= xz - cz + b \end{aligned} \tag{6}$$

Starting with initial values at (1, 1, 1) and using parameters a = .55, b = 2 and c = 4 he obtained the computer-drawn stereoscopic diagram shown in Figure 7. The complex pattern he called "Screw Chaos".

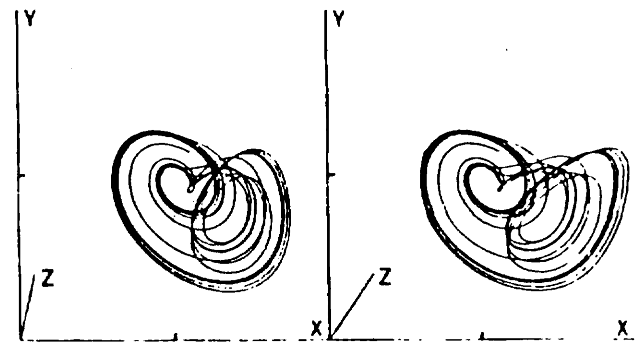


Figure 7: Stereoscopic Drawing of "Screw" Chaos Source: Rossler (1976, p.1668)

3.4 Strange Attractors and Instability

The set of points ultimately visited, or approached asymptotically by a dynamical system is called an *attractor.* Attractors like those discovered by Lorenz are called *strange*, a term coined by Ruelle and Takens (1971). Although quite a lot is now known about these attractors, their complexity is still a cause for wonder.

Certainly, they contain a moral of great importance and long emphasized by practitioners of system dynamics, namely, that nonlinear feedback systems are far richer than linear ones and that an understanding of future behavior of such systems must somehow be obtained from a description and analysis of structure; inference of future behavior based on past observation, especially of short time-series of observations made at discrete intervals, can *not* be relied on.

4. IRREGULAR ECONOMIC GROWTH CYCLES

4.1 Discrete Time Again

So far as I know no economic models of continuous time chaos have as yet been developed. It seems to me clear, however, that they soon will be. This conjecture is based on the intimate relationship between discrete and continuous time dynamics that has characterized the development of the theory presented so far. This intimate relationship was brought out in the beginning in Lorenz's original work. Thus, Lorenz, in an attempt to illustrate deterministic, dynamical irregularity in the simplest possible way observed.

> . . . the exact integration of a system of differential equations over a chosen interval of time determines a system of difference relations which is exactly equivalent to the original equations. When the original equations are non-linear, the equivalent difference equations generally cannot be written in finite form in terms of familiar analytic functions. The existence of the difference equations is assured, however, by the existence of solutions of the differential equations.

We therefore lose no generality, in choosing an arbitrary system of equations to illustrate the problem of deducing the climate, if we choose a system of difference equations instead of differential equations. The alternative methods of attack are still available, and they still possess their distinctive characteristics.

In the interests of economy, we shall seek the simplest possible system of nonlinear difference equations . . . Lorenz (1964a, p.3).

The equation Lorenz chose as the simplest was equivalent under a linear transformation to equation (3) which I arrived at in an analogous way from the Robertson-Williams Cobweb Model. This equation has the same type of quadratic nonlinearity possessed by the continuous time equations of forced dissipative flow (5) of which Lorenz's and Rosslers equations are examples, and because the equation is discrete, the chaos phenomenon emerges, as we have already seen in Figure 2.

4.2 The Li-Yorke Theorem

Difference equations are examples of "iterated maps" which can also be obtained from differential equations by considering the solutions of the latter at discrete (say "unit") intervals of time or at points where they intersect a transversal plane (Poincaré section). Their study is therefore fundamental to the analysis of dynamical systems of continuous or discrete time as shown by Smale (1967). In 1975 Li and Yorke used Smale's famous "horseshoe" technique to derive sufficient nonlinearity conditions for the existence of irregular trajectories in dynamic models.

Consider a real-valued, continuous function θ that depends on a vector of parameters, say π, that maps an interval J into itself. Such a function generates a difference equation

(7) $x_{t+1} = \theta(x_t; \pi) \equiv \theta(x_t)$,

whose behavior depends on the parameters of π. If for some value of these parameters there exists a point in J, say x^c, such that

(8) $\theta^3(x^c) \leq x^c < \theta(x^c) < \theta^2(x^c)$,

where $\theta^1(x) \equiv \theta(x)$, $\theta^2(x) \equiv \theta(\theta(x))$, and $\theta^{n+1}(x) \equiv \theta(\theta^n(x))$, then Li and York showed that

A. there exist cycles of every order in J. (That is, for every n, n = 1, 2, 3, . . . , there exist points in J satisfying $x = \theta^n(x)$).)

B. there exists an uncountable set S in J such that all trajectories with initial conditions in S remain in S and

B1. every trajectory in S wanders arbitrarily close to every other one,

B2. no matter how close two distinct trajectories in S may come to each other, they must eventually wander away,

B3. every trajectory in S wanders away from a cycle of any order in J, however close it may approximate one for a time.

The functions θ^n are the "iterates" of θ and the condition (8) may be thought of as a sufficient nonlinearity condition.

In Figure 8 a point x of the theorem is a point like "a" which maps into "b" which maps into "c". These successive

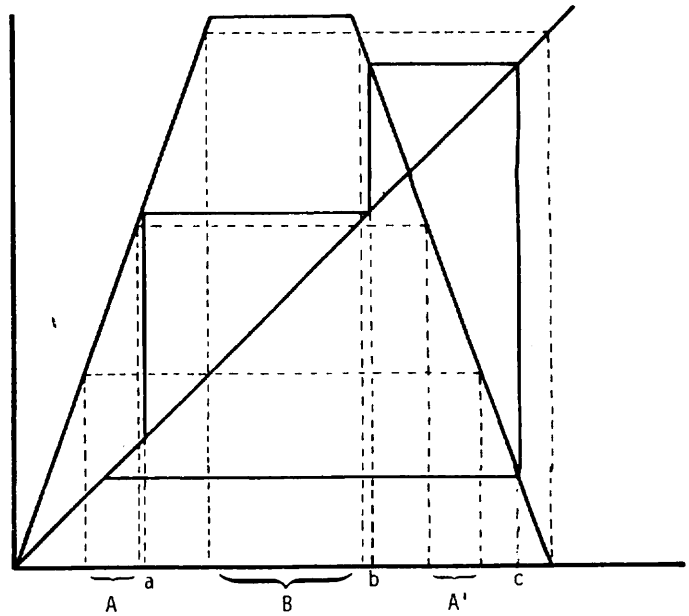


Figure 8: The Li-Yorke Sufficient Overshoot Conditions

increases are followed by the third iterate of "a" that overshoots the equilibrium or fixpoint $\bar{x}$ and falls to or below "a". In this example points in the set B map into the zero fix-point. The sets A and A' map into B and by similar construction a doubly infinite sequence of intervals can be found mapping into zero. Yet, the chaos theorem is satisfied: sandwiched in among these intervals exist cycles of every order and an uncountable, scrambled set of chaotic trajectories more or less like the ones shown in Figure 3!

The proof provided by Li and Yorke is of considerable interest since it helps provide an intuitive feel for how complex behavior can arise and because it can be readily extended to the case of n-variable difference equations and to set-valued dynamical systems of the kind arising in the simulation models described in Section 2. At this point we want to show how the theorem can be used to establish the existence of chaotic trajectories for simple recursive economies such as our Robinson-Williams Cobweb model.

Elsewhere, I have shown how, as the extent of the market (represented by the parameter m in equation (3)) increases, the overshoot condition eventually is satisfied. One can easily get the idea by observing that the maximizer of (3) which is $x^* = ½$ is unchanged as m increases. The maximum, however, $x^{**} = m/4$, does increase with m. Then the "overshoot" $f(x^{**}) = m^2(4-m)/16$ must be less than the preimage of x^* as m gets close enough to 4. Actually, the overshoot is satisfied for m = 3.86+ and for high enough odd iterates the overshoot parameter approaches 3.57+ which is the established threshhold for chaos for equation (3).

4.3 A Neo-Classical Growth Model

In another study Day (1982) I have applied the Li-Yorke theorem to a nonlinear discrete-time analog of the Tinbergen-Solow-Swan model of economic growth. The latter boils down to the equation

$$(9) \quad k_{t+1} = \min\left\{(1+\rho)k_t, s(k_t)f(k_t)\right\}/(1+\lambda)$$

where s(k) is the propensity to save, f (k) is the per capita production function, ρ a bound on the rate of investment and λ the net population growth rate. The nonlinearity required to produce the chaos generating overshoot can be induced by nonlinearity in either s(k) or f(k). I won't reproduce these results here.

However, a simulation of (9) is instructive for it shows how irregular growth cycles somewhat like those observed in GNP can be generated by the simplest possible discrete-time model. This is done in Figure 9.

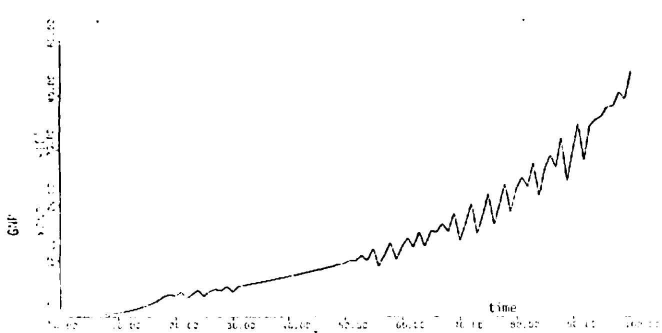


Figure 9: Irregular Growth Cycles in GNP: The "Neoclassical Growth Model", Source: Day (1982)

5. CONCLUDING OBSERVATIONS

Emphasized throughout our discussion has been the chaos generating property of nonlinear difference and differential equations. The kind of nonlinearity required for the emergence of such irregular fluctuations is present in the simplest micro- or macro-economic models using discrete time. But these nonlinearities are analogous to those that arise in continuous-time models of turbulence in forced, dissipative systems of dimension 3 or more. It seems likely, therefore, that the continuous time phenomenon of strange attractors will be found in the kind of economic models now being investigated by the system dynamics school, since the features of forced dissipative systems would seem to be relevant for the study of economic phenomena. That in fact is the basic idea behind the Prigogine-Allen models of spatial demoeconomics mentioned earlier in the text. Indeed, a standard system dynamics analysis of that work would be an interesting exercise and would help bring out in dramatic fashion insights of Forrester mentioned in his classic work, namely, the counter intuitive nature of many results obtained in nonlinear dynamics, and the sensitivity of solutions to initial conditions.

The Li-Yorke existence theorem uses a technique devised by Smale for the study of continuous time dynamical systems which exploits ideas going back to Poincaré, who already was aware of the complex dynamics in the three body problem of classical mechanics. This is not the place to go into those techniques, but suffice it to say, they further establish the intimate relationship between discrete and continuous-time systems as brought out in more detail in a more recent contribution by Smale (1976).

Thus, I think we can be confident that the kind of economic chaos illustrated in the course of our discussion is not an artifact of discrete time but is highly likely to surface as more experience with system dynamics models accumulates. If this is to happen, however, it will be important for system dynamicists not to mistake chaos for transient motion which has not had time to die out.

Thus I forecast with some confidence, but with some irony, that we will soon be well on our way to an endogenous theory of economic irregularity and a rigorous explanation of why prediction is so difficult in this field. If after testing such a theory against the facts it is found to have compelling empirical support, then it will be time to consider the ramifications on how we think about economic policy. It seems to me these ramifications are likely to be profound.

From the point of view of economic research methodology it seems also that profound changes in viewpoint may result, for if nonpredictability is intrinsic to economic structure, then theories can hardly be blamed for failing in this regard. Attention then would have to be shifted to qualitative prediction and to the plausibility of underlying assumptions both of which provide valid empirical grounds for preferring one theory over another.

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