A GENERIC POSITIVE FEEDBACK STRUCTURE

R. Joel Rahn, Faculte des Sciences de L'Administration, Universite Lava, Ste-Foy, Quebec, Canada

The author is an associate professor in the Department of Operations and Decision Systems at Universite Laval where he has been teaching system dynamics, simulation and operations management since 1974. The work reported in this paper was performed while the author was on sabbatical leave at the System Dynamics Group at M.I.T. It is a pleasure to acknowledge the many fruitful discussions with G.P. Richardson during the development of many of the ideas presented here. Responsibility for any errors or omissions is entirely my own.

ABSTRACT

Analysis of some typical structures found in system dynamics models of organizational and economic systems has shown that it is possible to define a constant of the motion associated with positive feedback loops having unity gain, and that such a constant is a sensitive indicator of the under-lying dynamic nature of the multiple loop structure. In this paper, a general treatment of a class of positive feedback structures is developed in which the unity-gain examples appear as singular cases. The treatment includes development of a canonical form of the structure, transformation of the structure to reduce the order of the system, discussion of the eigenvalues of the linearized structure, some properties of open-loop and closed-loop step and pulse gains, general formulas for the constants of the motion, an additivity property of the structure and a brief discussion of the effects of stochastic influences on the generic structure.

1. INTRODUCTION

To understand the relationship of structure to behaviour, one can classify structures according to some idea of their generic nature or 'genericity', and then categorize the behaviour of these structures under different conditions. In introductory courses this program often takes the form of presenting simple one-level models of growth, adjustment to equilibrium and S-shaped growth, followed by discussion of two-level models whose behaviour encompasses growth, decay and oscillations. The generic nature of these models resides in large part in their ubiquity, simplicity and transparency.

> **Ubiquity**: All dynamic models can be described in terms of inter-connected one-level sub-models.
>
> **Simplicity**: A one-level model is the simplest dynamic structure possible; a two-level model is the simplest structure that can show oscillatory behaviour.
>
> **Transparency**: No more than three parameters are necessary to characterize the structure and the behaviour of linear one- and two-level models.

The object of this paper is to present a generic structure which may be composed of an arbitary number of levels and to show how its behaviour may be determined relatively simply by means of its open-loop step gain. As well, we develop a very general or canonical form for the structure and we discuss various aspects of the structure such as its eigenvalues, closed-loop step and pulse gains, effects of stochastic influences on the structure and the definition of linear combinations of the levels which are constant when the open-loop step gain is unity. The discussion will be largely abstract. Readers who are interested in concrete examples are invited to refer to a companion paper.[1]

1.1 Generic Structure

A generic structure consists of many loops connected in a stable and meaningful way. Stability in this context means that links between elements are maintained over some range of values of the levels in the loops composing the structure. The criterion of meaningfulness serves to focus attention on those structures that represent organizational or social relationships found in functioning systems and to eliminate randomly generated structures and pathological cases. The concept of generic structure has been proposed informally as a means of generalizing the understanding of some multi-loop structure to other similar situations. Candidates for this designation would surely include the Production-Distributions system,[2] the Commodity Cycles model[3] (probably the first reference to the concept under the name generic structure), and several renewable, and non-renewable resource and ecological models[4]. The concept of generic structures is an important contribution to the conceptualization of complex systems. The generic nature of a structure should help the analyst make sense of the wide variety of information used to conceptualize the basic problem-generating structure. Careful use of these structures is necessary to avoid the 'hammer in search of a nail' syndrome. However, they have proven their pedagogical value in permitting students from widely different back-grounds to achieve a degree of facility in communicating their understanding of system structure and behaviour to equally disparate audiences.

Another use for generic structures is for policy prescription or system design. An understanding of the behaviour of somewhat more complex structures than the simple one- and two-level models studied in introductory courses should be useful as a guide to modifying system structure to achieve some dynamic performance goal. The idea is to provide a catalog of well-understood structures so that the analyst can focus on those elements of structure that are most important for his understanding of behaviour. The list of generic structures which are thoroughly and succinctly analyzed is still rather limited and there remains much to be done to characterize the relationship of structure to behaviour in those few structures that are commonly viewed as generic. The analysis here should contribute to improving the ability of the analyst to understand the effects of his modifications of the system in the search for improved policies.

The feedback loop is the fundamental unit of analysis in system dynamics. Yet it has proven difficult to develop an intuitive understanding of the behaviour of systems of feedback loops by reference to the behaviour of the component loops. The difficulties are evidenced by a number of efforts to avaluate casual loop diagrams as useful aids to analysis[5,6]. The quality of transparency begins to be strained even for the behaviour

of two-level models as witness the use of the somewhat recondite concept of phase-shift to understand oscillatory behaviour[7].

In this paper, we focus on the behaviour of a generic structure rather than its component loops. A great economy of thought is realized since the behaviour of the generic structure can be characterized by a relatively small number of parameters whose meaning is intuitively clear to those responsible for managing the system.

The generic structure of interest in this paper is organized around a major positive feedback loop. Before describing this structure, some definitions are introduced to specify more clearly some of the concepts that will be used throughout the paper It would seem unnecessary at this late stage in the development of the field to define what is meant by a feedback loop, or briefly, a loop. However, experience has shown that some of the difficulties in communicating an understanding of the behaviour of unity-gain loops, for example, are due to sloppy definition of the structure under discussion. These definitions are offered in order to minimize misunderstanding of the relationships between structure and behaviour to be developed in later sections.

> Major loop – a set of levels in which each level in the set is linked to one succeeding in the set. Each level influences only one other level and its influenced by only one other level in the set.
>
> Minor loop – a level and a relationship such that the level influences itself without influencing other levels.
>
> Sub-major loop – a subset of the levels in a major loop in which each level in the subset is linked to one succeeding level in the subset. Each level in the subset influences only one other level and is influenced by only one other level in the subset.

These definitions are illustrated in Figure 1 by a casual loop diagram derived from the Salesman-Backlog-Delivery Delay structure in the classic 'Market Growth' paper[8] which describes another possible entry in the library of generic structures. In the figure, the only levels for purposes of this discussion are Salesmen, Backlog and Delivery Delay Average. We emphasize here the distinction that is made between **loops** and **structures**. Loops are components of structures. They may be of some dynamic interest in their own right, but the reason for studying structures and particularly generic structures is to gain some understanding of how the interactions among loops connected in 'generic' ways explains behaviour.

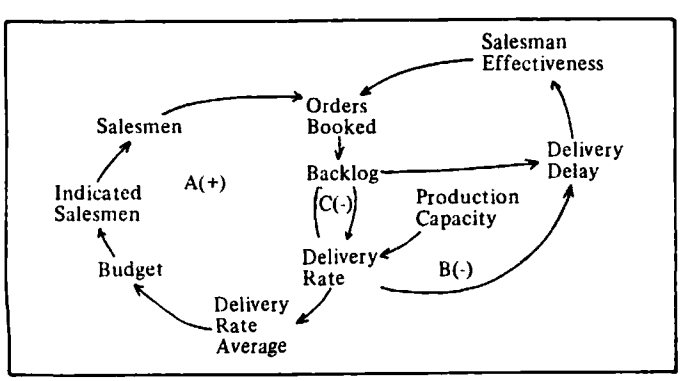

Figure 1: Examples of loops – A: Major, B: Sub-Major and C: Minor.

2. POSITIVE MAJOR LOOPS

To introduce some notation we start with a brief discussion of an important component of the generic structure to be introduced later. A positive major loop containing more than one level is characterized by links of positive polarity (or links of negative polarity occuring in pairs) from each level to the succeeding level. In analytic terms this means that the derivative of the input rate to the succeeding level with respect to the preceding level is positive (or there are pairs of derivatives that are negative). A major loop can be described by means of an operator-matrix formalism as follows. We let the levels be represented by x_i, i=1, ...,n. The levels are numbered in succession around the major loop starting at any level. The influence of level i-1 on its successor, level i is described by a rate function which is a general function of level i-1, $a_{i,i-1}(x_{i-1})$. To close the loop, the last level, x_n influences its successor, x_1 through the function $a_{1n}(x_n)$. Starting with the first-numbered level and working forward through the predecessors to the last-numbered level, we can represent a major loop by the set of equations:

$$\begin{aligned} \dot{x}_1 &= a_{1n}(x_n) \\ \dots &\quad \dots\dots\dots\dots \\ \dot{x}_i &= a_{i,i-1}(x_{i-1}) \\ \dots &= \dots\dots\dots\dots \\ \dot{x}_n &= a_{n,n-1}(x_{n-1}) \end{aligned} \tag{1}$$

Note that we ignore all exogenous inputs in order to concentrate on the endogenous structure of the system. Further, we remark that the levels in system dynamics models are usually defined to be non-negative quantities. We will assume that such is the case and will further specify that for all of the functions $a_{i,i-1}$,

$$a_{i,i-1}(x_{i-1}) = 0 \text{ when } x_{i-1} = 0. \tag{2}$$

We introduce a ring symbol (o) to represent functional dependence so that

$$a_{i,i-1}(x_{i-1}) = a_{i,i-1} o x_{i-1}$$

Then the above set of equations can be written as a vector differential equation for $x = (x_1, \dots, x_n)^t$:

$$\underline{\dot{x}} = \begin{bmatrix} 0 & \dots & \dots & a_{1n}o \\ a_{21}o & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 \dots & a_{i,i-1}o & 0 \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & a_{n,n-1}o & 0 \end{bmatrix} \underline{x} \tag{3}$$

We assume that the functions $a_{i,i-1}$ have non-zero derivatives of some order. The loop is positive if and only if the product of the first non-zero derivatives is positive. For example, in the usual case that the first derivatives are all non-zero, the loop is positive if

$$a'_{21} \cdots a'_{i,i-1} \cdots a'_{n,n-1} a'_{1n} > 0$$

where primes (') indicate differentiation with respect to the appropriate state variable (whose index is given by the second index of the function).

No general solution of equation (3) is known. However, as is usual in dealing with nonlinear systems, some general statements can be made about the linearized version of the equation, in which the functions $a_{i,i-1}$ are replaced by their derivatives $a'_{i,i-1}$ evaluated at some point in the state space. As has been shown elsewhere[7], the general solution of the linearized version of equation (3) includes oscillating and decaying terms. For <u>any</u> system with more than one level, the eigenvalues of the (linearized) model are in general complex so that one expects growing or damped oscillations as the characteristic behaviour modes. However the long-term, persistent behaviour is dominated by the eigenvector whose eigenvalue has the algebraically largest real part. Except for very special initial conditions, the relevant eigenvalue in the solution of the linearized form of equation (3) will be positive and real. Thus the tendency for the system in equation (3) is to grow or decline without limit.

Real systems do not grow without limit. The more usual case is to find a positive feedback loop linking a set of levels which are also included in one or more negative loops. When such a negative loop is a minor negative loop, the unbounded growth of any positive level due to unbounded input from the preceding level in the major loop is counteracted to some extent by an unbounded growth in the outflow from the given level. The local control by the minor loop can compensate for the unbounded growth of the major loop and may precent that growth from being unlimited. A simple case of a multi-loop structure composed of a positive major loop with negative loops included to prevent unbounded growth is given by having each level in the major loop controlled by a minor negative loop.

In the operator-matrix formalism introduced above, we represent the minor negative loops by functions

$$-a_{ii}(x_i) = -a_{ii} o x_i \tag{4}$$

These functions may be non-linear but they must be such that:

- the first non-zero derivative of the function $a_{ii}o$ is positive;
- the limiting value of the function as x_i tends to zero is zero.

The first condition guarantees that the minor loop is indeed negative according to equation (4). The second condition guarantees that there is no outflow from level x_i when x_i is zero. The functions a_{ii} correspond to generalized, first-order delay structures for each level as emphasized by the explicit use of the negative sign. Using the operator-matrix notation, we can write the model as in Figure 2.

$$\dot{\underline{x}} = \begin{bmatrix} -a_{11}o & & & a_{1n}o \\ a_{21}o & -a_{22}o & & \\ & a_{i,i-1}o & -a_{ii}o & \\ & & a_{n,n-1}o & -a_{nn}o \end{bmatrix} \underline{x}$$

Figure 2: General Positive Major Loop with Nonlinear Minor Loops.

2.1 Linearizarion of the General Model

Not a great deal can be said about the characteristics of the non-linear model shown in Figure 2 without some further motivation. Thus we turn to a linearized version that has somewhat more familiar components (delay times, growth factors) found in elementary structures. Since system dynamics models can be formulated as piece-wise linear vector differential equations, we do not lose much practical generality in the arguments that follow.

When the delay functions a_{ii} are linear, we can introduce the delay time, d_i, explicitly.

$$\dot{\underline{x}} = \begin{bmatrix} -1/d_1 & & & a_{1n}o \\ a_{21}o & -1/d_2 & & \\ & a_{i,i-1}o & -1/d_i & \\ & & a_{n,n-1}o & -1/d_n \end{bmatrix} \underline{x}$$

Figure 3: N-level, Non-linear, Positive Major Loop with Linear, Negative Minor Loops.

$$a_{ii} o x_i = (1/d_i) x_i \tag{5}$$

and the model in Figure 2 can be written as in Figure 3.

To linearize the functions forming the major loop, we can expand each function in a Taylor's series about some value, x_{i-1}, of the input variable x_{i-1}:

$$a_{i,i-1}(x_{i-1}) = a_{i,i-1}(X_{i-1}) + (x_{i-1} - X_{i-1}) a'_{i,i-1}(X_{i-1}) \tag{6}$$

with the notion

$$a'_{i,i-1}(X_{i-1}) = da_{i,i-1}/dx_i(X_{i-1}) \tag{7}$$

We can write the linearized model as in Figure 4. Terms consisting of the constants $a_{i,i-1}(X_{i-1})$ are omitted since they correspond to exogenous inputs.

$$\dot{\underline{x}} = \begin{bmatrix} -1/d_1 & & & a'_{1n} \\ a'_{21} & -1/d_2 & & \\ & a'_{i,i-1} & -1/d_i & \\ & & a'_{n,n-1} & -1/d_n \end{bmatrix} \underline{x}$$

Figure 4: Linearized N-level Positive Major Loop with Negative Minor Loops.

The equations in Figure 4 correspond to a cascade of first-order material delays linked in a major loop which is positive because the product of the coefficients in the links from one level to the next are positive, i.e.

$$a'_{21} \cdots a'_{i,i-1} \cdots a'_{n,n-1} a'_{1n} > 0$$

It is easy to see that the same structure corresponds to a linked set of information delays if we write

$$a'_{i,i-1} = (1/d_i) b_{i,i-1} \tag{8}$$

so that the links between succeeding levels are given by the coefficients $b_{i,i-1}$ and each level is an exponential smoothing of its predecessor. We can represent a structure consisting of an arbitary succession of material and information delays by re-defining, where necessary, the coefficients linking any two levels in this way. The generality of the structure demonstrated here is a strong argument in favor of its designation as a generic structure.

A further simplification of the dynamic information can be made by rewriting the system matrix as the product of two matrices, as shown in Figure 5.

$$\dot{\underline{x}} = \begin{bmatrix} 1/d_1 & & & & \\ & 1/d_2 & & & \\ & & \cdot\ 1/d_i & & \\ & & & \cdot\ 1/d_n \end{bmatrix} \begin{bmatrix} -1 & & & & d_1 a'_{1n} \\ d_2 a'_{21} & -1 & & & \\ & \cdot\ d_1 a'_{i,i-1} & \cdot\ -1 & & \\ & & \cdot\ d_n a'_{n,n-1} & \cdot & -1 \end{bmatrix} \underline{x}$$

Figure 5: Linearized Positive Major Loop, Negative Minor Loops with Delay Coefficients Incorporated in the Level-to-level Links.

Defining the diagonal matrix component as

$$D^{-1} = \mathrm{diag}\,(1/d_i,\ i=1, \ldots, n) \tag{9}$$

and the system matrix component, A, as the second matrix on the right-hand side in Figure 5, the vector equation in Figure 4 can be written more compactly as

$$\dot{\underline{x}} = D^{-1} \underline{A} x \tag{10}$$

3. CANONICAL FORM OF THE NON-LINEAR GENERIC STRUCTURE

We now turn to the derivation of a special, abstract form of the generic structure which represents in a simple way all of the essential dynamics and which is of the same form for linear and non-linear models. The fact that this form is at the heart of the generic structure impels us to call it the **canonical form** of the generic structure. While no new dynamic information is revealed by the canonical form, it motivates some of the succeeding discussion and emphasizes the special character of the **unity-gain condition** which is treated in a companion paper[1].

It is a fairly natural step to define a canonical form of the generic nonlinear structure which corresponds to the vector equation

$$\dot{\underline{x}} = \underline{A} x \tag{11}$$

where A is the generalization to the nonlinear case of the form of the right-hand matrix shown in Figure 5, or equivalently, the form of the matrix A in equation (10). Although we have written equation (11) using a state vector $\underline{x}$, we emphasize that there is no necessary relationship between this state vector and the vector used up to equation (10). We concentrate on the form of the operator-matrix A; i.e., with non-zero elements below the diagonal except for the 'corner' element, and -1 on the diagonal. Even in the linear case there is no transformation of the state space which allows us to re-write the equation in Figure 5 without the diagonal matrix D or its inverse. We will show how to express the generic nonlinear structure in terms of a diagonal matrix operator and the canonical matric operator. The resulting structure has all the parameters of dynamic interest concentrated in the major loop coefficient functions and this makes it easier to develop a nonlinear generalization of the open-loop step gain.

In the linear analysis, multiplication of the right-hand side of each equation by the corresponding delay time leads to a system matrix with -1 on the diagonal. This operation is equivalent to inverting the linear function.

$$(1/d)_i o x_i = x_i / d_i \tag{12}$$

For the general case of non-linear delays, we achieve the same result by operating on the right-hand side of each equation with the inverse of the delay function $a_{ii}o$. These operations are valid if the delay function has an inverse, e.g., if the first derivative is non-zero in the domain of operation of the level. The inversion operations can be performed by writing the identity matrix as the product of factors

$$\mathrm{diag}(a_{ii}o)\mathrm{diag}(a_{ii}^{-1} o)$$

where

$$a_{ii} o a_{ii}^{-1} o x_i = 1 \tag{13}$$

The resulting form of the non-linear structure is shown in Figure 6.

Replacing the off-diagonal terms by

$$A_{i,i-1}o = a_{ii}^{-1} o a_{i,i-1} o, i=2, \ldots, n;\ A_{1n}o = a_{ii}^{-1} o a_{in} o \tag{14}$$

$$\dot{\underline{x}} = \begin{bmatrix} a_{11}o & & & \\ & \cdots & & \\ & & a_{11}o & \\ & & & \cdots \\ & & & & a_{nn}o \end{bmatrix} \begin{bmatrix} -1 & & & & a_{11}^{-1} o a_{1n} o \\ & \cdots & & & \cdots \\ & a_{11}^{-1} o a_{i,i-1} o & -1 & & \\ & & \cdots & & \cdots \\ & & & a_{nn}^{-1} o a_{n,n-1} o & -1 \end{bmatrix} \underline{x}$$

Figure 6: Decomposition of Non-linear Generic Structure into Diagonal and Canonical Components.

We have the matrix operator Ao in the desired canonical form corresponding to the linearized system matrix A in Figure 5. Note that the operators $A_{i,i-1}o$ are the non-linear analogues of the coefficient $b_{i,i-1}$ introduced above in the demonstration of the generality of the linked material and delay structure. The dynamics is determined essentially by the effect of the composite operator

$$A_{n,n-1}oA_{n-1,n-2}o \ldots A_{i,i-1}o \ldots A_{21}oA_{1n}o \cdot 1 \qquad (15)$$

which translates the effect of an impulsive change in x_n as it propagates around the loop. The first term, read from right to left, is the result of the impulsive change in x_n as it affects first x_1, then $x_2, \ldots, x_{i-1}, \ldots x_{n-1}$ and ultimately x_n again. An impulsive change in x_n is transformed by the system into an input rate to x_n that depends on the gain around the major loop. The second term (-1) is the effect of the delay operator on a change in x_n, i.e., the delay operator changes an impulsive input into an outflow rate of equal magnitude (initially). Subsequently, the unity delay operator maintains the outflow rate equal to the quantity in the level x_n. If the input to x_n is a unit step, expression (15) shows that the behaviour of the system is determined by whether the composite operator amplifies or dissipates the energy in the step input; i.e., whether the result of the nonlinear operator is greater ir less than the unity operator. We will see in section 3.3 below that the non-linear component in expression (15) is a generalization of the open-loop step gain. Having shown that the canonical form can be developed from the general non-linear case, we turn now to a study of some interesting aspects of this form.

3.1 Reduction of the Order of the Major Loop

In some cases it is possible to reduce the effective number of levels of interest in the major loop even before linearizing. For example, a level x_2 may be created by smoothing the outflow rate from the level x_1. This occurs in the Salesman-Backlog loop in Figure 1 where Delivery Rate Average is a smoothed value of Delivery Rate. In terms of our generic model, we have in the notation of Figure 2.

$$a_{11}o = a_{21}o \qquad (16)$$

By adding the first two rows of the operator-matrix, we can eliminate the entry in the first column of the second row. This is achieved by multiplying the equation in Figure 2 by the matrix.

$$T = \begin{pmatrix} 1 & 0 & .. & .. & 0 \\ 1 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & 0 .. & 0 \\ 0 & .. & .. & .. & 1 \end{pmatrix}$$

which gives:

$$\begin{array}{l} \dot{x}_1 = \\ x_1 + x_2 \\ \dot{x}_3 \\ .. \\ \dot{x}_n \end{array} \begin{pmatrix} a_{11}o & 0 & .. & .. & .. & a_{1n}o \\ 0 & -a_{22}o & & 0 & .. & a_{1n}o \\ 0 & a_{32}o & & -a_{33}o & 0 & ..0 \\ 0 & .. & .. & .. & & 0 \\ 0 & .. & .. & a_{n,n-1}o & & -a_{nn}o \end{pmatrix} \underline{x} \qquad (17)$$

We see that the levels $(x_1+x_2), x_3, \ldots, x_n$ are in a positive major loop of the generic type we have been discussing, while x_1 is apparently driven by the output of the loop, x_n, and responds with a delay given by $-a_{11}ox_1$. In some sense we can consider the major loop to be composed of only the last n-1 equations. Unfortunately, we cannot make this observation more precise. Even in the linear case, if we transform the differential equation by using the matrix T, the generic 'shape' of the resulting matrix is lost. However, it is evident that this kind of reduction of the order occurs in any vase where an outflow rate is delayed. The commonly occuring cascade of first-order material delays in one such case. The formal reduction in order of the major loop structure justifies the usual, intuitive practice of ignoring the effects of delays when discussing long-term behaviour.

3.2 Transformation to Alternative Forms

It is a simple matter in the linearized case to show alternative forms that are equivalent to the canonical form of Figure 5. These forms are simply different ways of writing the original canonical form. Suppose we transform the vector of level variables by multiplying it on the left by a matrix of constants, T, to get new variables, $T\underline{x}$, which are linear combinations of the original level variables. The differential equation can be re-written in terms of these new state variables. In the nonlinear case, there are no general, intuitive results for such an operation. Again we look to the linear case for guidance, and we begin by linearizing equation (11) so that the operator-matrix A is replaced by the matrix of derivatives of the coefficients functions of A, namely, A'. Thus we consider the equation

$$\dot{\underline{x}} = A'\underline{x} \qquad (11')$$

To determine the equation for the new variables, we multiply equation (11') by the matrix T on the left and then use a form of the identity matrix that is convenient, namely

$$I = T^{-1}T$$

to get

$$T\dot{\underline{x}} = TA'T^{-1}T\underline{x} \qquad (18)$$

so that the derivative of the new vector of state variables appears on the left-hand side and the new vector itself is multiplied by the transformed matrix $TA'T^{-1}$. It is well-known that linear transformation of linear differential equations has no effect on the dynamic character of the structure[9]. The effect of a transformation is merely to re-define the level variables. Presumably the original levels have been chosen for their real-world significance so that new combinations of levels should correspond to new concepts relevant to the system under investigation. Ideally, we should be able to interpret the effects of a transformation, and what it reveals about the system's structure and behaviour in terms familiar to those responsible for operating the system. In abstract analyses such as the present one, transformation may help to simplify or generalize the analysis and bring out subtleties of the dynamic character of the system that otherwise would require much simulation to reveal.

One way of re-writing the system matrix that is of interest is the transpose of the given canonical form. The transpose

of A' (denoted A'^T) can be created by transforming the original state vector by the matrix

$$T = \begin{pmatrix} 0 & .. & .. & 0 & 1 \\ 0 & .. & .. & 1 & 0 \\ 0 & .. & 1 & 0 & 0 \\ 0 & .. & .. & .. & 0 \\ 1 & 0 & .. & .. & 0 \end{pmatrix} \tag{19}$$

This matrix is its own inverse so that

$$TA'T^{-1} = TA'T = A'^T \tag{20}$$

The result of transforming $\underline{x}$ with T is to reverse the ordering of the levels so that

$$T\underline{x} = (x_n, \ldots, x_i, \ldots x_1) \tag{21}$$

is the state vector operated on by A'^T to give the dynamics which evidently are not affected by what are nominal changes only.

Another form of the system matrix is superficially similar to the transpose. With the original ordering of the levels $x_1, \ldots, x_i, \ldots x_n$, it represents a major loop traced in the opposite direction. That is, starting at x_n, the causality runs from x_n to x_{n-1} all the way back to x_1. In any given set of levels, these two forms, the canonical form of Figure 5 and the transpose form, with the original ordering of the levels maintained, represent the only two major loops that can exist. In a real system, these two loops will have different meanings and often only one will exist in fact. A pedagogical example of an urban housing, business structures and population model[10] has both major positive loops with minor negative loops and hence represents, superficially, a composition of two generic structures of the type under discussion here. We can represent such a composition by splitting the system matrix into two parts, each of which is of generic type, with one part having a matrix of the same shape as A'(with a non-zero element in the 'north-east' corner), and the other a matrix of the same shape as the transpose of A' (with a non-zero element in the 'south-west corner). However, the two major loops cannot be treated separately, nor can one analyze one of the loops as a perturbation added to the other. The interconnections provided by the sub-major loops which result inevitably from the presence of both major loops are not insignificant. One might summarize these comments on the transformation of the linearized generic structure and the addition of the direct and transpose forms by the verbal formula.

> The generic nature of the structure under discussion is preserved under linear transformation, but is not additive.

It will be shown below that a suitable definition of the meaning of 'addition' of the generic structure under discussion allows us to specify cases in which two or more such structures can be added in an intuitively appealing though non-arithmetic way.

3.3 Gain

Each major-loop coefficient in the canonical form of Figure 5 represents the 'gain' acquired by a variable as it is transformed in the transmission from one level to the next. The concept of gain as used by system dynamicists is often ambiguous. For purposes of this paper, the only definition of gain we will use is the 'open-loop step gain'[11] and analogous measures for closed-loop and pulse-input cases.

The input rate to a level must have dimensions [level/time]. In physical terms, the gain factors represent the transformation of the dimensions of one level into those of another as well as any changes in magnitude that are imposed. Conserved flows are represented by a cascade of material delays and no change in dimensions is imposed by the loop coefficients. The outflow from one level is the inflow to the next so the loop coefficient in the canonical form is the ratio of the delay time of the given level to the delay time of the preceding level. Non-conserved flows of information permit changes in dimension as well as magnitude. Changes in magnitude represent one aspect of the distortion of information that can occur as it propogates through the system[12].

Several interesting aspects of the system can be simply described in terms of the gain. Let the product of the loop coefficients in the system matrix A of Figure 5 be g, so that

$$g = \left(\prod_{i=2}^{n} a'_{i,i-1}\right) a'_{1n} \left(\prod_{i=1}^{n} d_i\right) \tag{22}$$

Then g is the **open-loop step gain (OLSG)** around the major loop of the linearized generic structure shown in Figure 4. This is shown by the following argument. The open-loop step gain is defined to be the limit as time goes to infinity of the change in the major loop feedback input to level x_1 as a result of a unit step change in the input to the level x_1. The designation 'open-loop' comes from considering the effect of the input step to propogate through x_1 to x_n and then back to the input to x_1 in what is conventionally called the open loop of the system. In the case of positive unity feedback, the open loop gain is the same as the feedforward gain which is defined to be the ratio, as time goes to infinity, of the change in the output of the system (measured as the change in the rate of change of x_1 due to x_n) to the change in the input (the unit step to x_1).

To specify the system in terms of a feedforward part and a feedback part, we consider the case of a unit step input to the level x_1. We construct the block-diagram representation for the system by taking the Laplace transform of the equation for each level x_i, i=2, . . .,n.

$$sx_i = a'_{i,i-1}x_{i-1} - x_i/d_i \tag{23}$$

or, solving for x_1 in terms of the preceding x_{i-1}

$$x_i = a'_{i,i-1}x_{i-1} /(s+1/d_i) \tag{24}$$

The Laplace transform of the equation for x_1 gives

$$x_1 = (a'_{1n}x_n + 1/s)/(s+1/d_1) \tag{25}$$

where the transform of the unit step input is 1/s. These equations can be represented in block diagram form as shown in Figure 7. The Laplace transform of the feedforward step response is

$$0(s) = a'_{1n}x_n(s) \tag{26}$$

and is simply the product of the Laplace transform of the unit step input and the Laplace transform of each delay and gain element on the feedforward path. Thus we have

$$0(s) = \frac{1/s(a'_{21} \cdots a'_{n,n-1} a'_{1n})}{\prod_{i=1}^{n} (s + 1/d_i)} \tag{27}$$

From the equality of the open-loop and feedforward gains when the feedback is +1 and from the final value theorem[13] we have the open-loop step gain

$$\text{OLSG} = \lim_{t \to \infty} 0(t) = \lim_{s \to 0} s0(s) = (a'_{21} \cdots a'_{n,n-1} a'_{1n}) \left(\prod_{i=1}^{n} d_i\right) = g \tag{28}$$

Note that in the case that some of the delays are information delays, the corresponding delay times do not determine the gain. Defining the coefficient $a'_{i,i-1}$ as in equation (8), we see that the delay time in the coefficient is cancelled by the delay time in the product of delay times in the above expression for the gain. Again we see a justification for the practice of ignoring information delays when assessing the long-term behaviour of a system.

The closed-loop step gain can be defined similarly as the limit, as time goes to infinity, of the change in the output of the closed loop due to a unit step change in the input. The closed loop is defined by the unity positive feedback of the feedforward output 0(s) to the input to x_1. We take the output to be $a'_{1n}x_n$ as for the feedforward or open-loop gain. To determine the closed-loop output we consider the block diagram of the system and trace back the source of the output as being due to the effect of each of the levels on the total input to x_1 which is $(a'_{1n}x_n + 1/s)$. Thus the feedforward operator is applied not only to the input step 1/s but also to the feedback signal, $a'_{1n}x_n$. The result is

$$a'_{1n}x_n = 0(s)\,(a'_{1n}x_n + 1/s) \tag{29}$$

$$= \frac{(a'_{1n}a'_{n,n-1} \cdots a'_{21})(a'_{1n}x_n + 1/s)}{\prod_{i=2}^{n} (s + 1/d_i)} \tag{30}$$

or

$$a'_{1n}x_n = \frac{(a'_{1n} \cdots a'_{21})1/s}{\prod_{i=1}^{n} (s + 1/d_i) - (a'_{1n} \cdots a'_{21})} \tag{31}$$

Then the closed-loop step gain (CLSG) is

$$\text{CLSG} = \lim x_1 = \lim s x_1(s)$$

$$\lim_{t \to \infty} \quad \lim_{s \to 0} = \frac{\left(\prod_{i=1}^{n} d_i\right)(a'_{1n} \cdots a'_{21})}{1 = \left(\prod_{i=1}^{n} d_i\right)(a'_{1n} \cdots a'_{21})} = \frac{\text{OLSG}}{1 - \text{OLSG}} \tag{32}$$

when the limit exists. The limit does not exist when OLSG > 1 since $sx_1(s)$ has poles in the right-half s-plane, corresponding to growing exponential terms in the solution. When OLSG=1, the limit does not exist since $sx_1(s)$ has a pole at s=0, corresponding to a non-zero derivative of $x_1(t)$ as t goes to infinity. Both of these conditions are sufficient to invalidate the use of the Final Value Theorem.

Further insight into the behaviour of the generic structure is furnished by considering the response to a **pulse** input to x_1. The Laplace transform of the pulse is 1 so that the open-loop pulse gain (OLPG) is

$$\text{OLPG} = \lim_{s \to 0} \frac{s(a'_{21} \cdots a'_{1n}).1}{\prod_{i=1}^{n} (s + 1/d_i)} = 0 \tag{33}$$

However, the closed-loop pulse gain (CLPG) is

$$\text{CLPG} = \lim_{s \to 0} \frac{s(a'_{21} \cdots a'_{1n}).1}{\prod_{i=1}^{n} (s + 1/d_i) - (a'_{21} \cdots a'_{1n})} \tag{34}$$

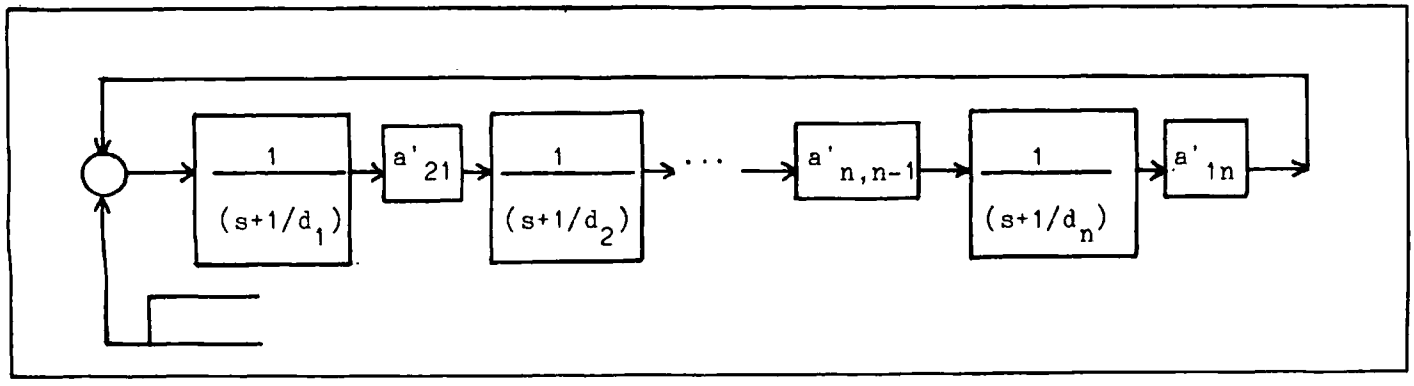

Figure 7: A Block Diagram Representation of the Linearized Generic Structure with Step Input.

$$CLPG = \lim_{s \to 0} \frac{s.\,OLSG}{(1-OLSG) + s(\sum_{i=1}^{n} d_i) + 0(s^2)} \quad (35)$$

For OLSG < 1, CLPG = 0. The response of a decay-type system to a pulse input is to absorb it and decay to zero. For OLSG > 1, the limit does not exist for the reasons stated above for CLSG. But for OLSG = 1 (the 'unity-gain' case[1]), the pole and zero at s=0 cancel and leave

$$CLPG = 1/\sum_{i=1}^{n} d_i \text{ for } OLSG = 1 \quad (36)$$

In the unity-gain case, a positive impulse is distributed around the loop so that each level is larger by an amount proportional to the reciprocal of the delay time around the major loop. The response to an impulse in a **unity-gain loop** is to change the **equilibrium** value of each level in the loop. This feature of the response of the unity-gain positive feedback structure inspires a metaphor for the system seen as a closed tube of compressible fluid, circulating around the major loop and transmitting the effects of exogenous inputs to each level in the loop[1].

Other features of the structure can also be written in terms of the open-loop step gain. It is simpler to work with the canonical form of the structure, but the following characteristics are relevant to the whole structure since the two forms are related by a simple (diagonal) linear transformation.

3.4 Eigenvalue Equation of the Linearized Model

As shown in Figure 5, the system matrix A captures all of the essential dynamic information about the linearized structure, while the diagonal component is a scale factor. This decomposition is seen more clearly when we compute the eigenvalues (or, equivalently the transfer function poles), s, of the system. The eigenvalue equation is

$$\det(D^{-1}A - sI) = \det D^{-1} \det(A - sD) = 0 \quad (37)$$

which reduces to

$$\prod_{i=1}^{n} (1 + sd_i) - (\prod_{i=2}^{n} d_i a'_{i,i-1}) d_1 a'_{1n} = 0 \quad (38)$$

It is evident that the diagonal matrix acts to scale the value of each eigenvalue according to the delay time d_i for each level. To simplify the succeeding discussion we define the eigenvalue scaled by the delay times to be

$$s_i = sd_i \quad (39)$$

The eigenvalue equation can be written simply as

$$\prod_{i=1}^{n} (1 + s_i) - OLSG = 0 \quad (40)$$

The fundamental dynamics of the structure is determined by the open-loop step gain of the system which is determined by two parameters

- the products of the delay times
- the product of the coeffieients of the relationships around the major loop.

This simple characterization of the basic dynamics is another reason to consider the structure to be generic. Although a general formula for s_i is not available, we can see the nature of the eigenvalues by considering the special case where all the delay times are equal so that

$$d_i = d, \text{ for } i=1, \ldots, n. \quad (41)$$

This case is a limiting case for the more usual situation that the delay times in a loop are not too disparate because all of the levels have been defined to correspond to system elements evolving over the same time horizon (or all levels have the same 'bandwidth'). In the case of equal delay times, the eigenvalues are given by

$$s_i = -1 + OLSG^{1/n} \quad (42)$$

where the n values are determined by the n (complex) nth-roots of the gain around the major loop. One of these nth-roots is real and positive because the gain is real and positive. The rest of the roots are distributed symmetrically about a circle in the complex plane, whose radius is given by the real nth-root of the absolute value of the gain. It is obvious from the above equation that the eigenvalues have negative real parts if and only if the absolute value of the second term is less than 1. Otherwise, there is at least one eigenvalue with real part non-negative. In the case that the real part is positive, the corresponding solution grows without limit. Thus this structure can show growth or decay, with or without oscillations. The addition of delays or smoothing increases the order of the major loop without changing the magnitude of the open-loop step gain. From equation (42), we see that as n increases the real part of the nth root of OLSG increases as cos(arg(OLSG)/n). Adding delays or smoothing tends to reduce the stability of this structure.

A somewhat more subtle property of the above structure appears if we consider the case where some of the levels are pure integrators and there is no delay term in the corresponding equation. Then the eigenvalue equation is

$$\prod_{i=1}^{n1} s_i \prod_{j=n1+1}^{n2} (1 + s_j) - OLSG = 0 \quad (43)$$

From this equation we see that n1 of the eigenvalues can be arbitrarily assigned and their product divided into the gain term to determine the remaining eigenvalues. The indeterminacy of the eigenvalues is a reflection of the non-physical nature of the model. Another aspect of the same un-realistic model is revealed by noting that the open-loop step gain of an integrator is not defined (is infinite). These properties show that the generic model, to be physically realistic, must have a delay term associated with each level in the major loop. The fact that the whole structure must be taken as defined in order to make sense as a model of a real process supports the contention that the structure is generic.

4. CONSTANTS OF THE MOTION

Having shown the generality of the canonical form and having produced some arguments in favor of the genericity of the full non-linear structure, we turn now to the derivation of certain supplementary variables, the so-called 'constants of the motion' which enable us to characterize the behaviour of the generic structure in a fundamental way. To simplify the analysis, we start from the linearized canonical form of Figure 5. We define the non-unity elements of the canonical matrix A by

$$A'_{i,i-1} = d_i a'_{i,i-1} \tag{44}$$

Noting that each column of A contains only two non-zero coefficients, we are lead to add the equations in such a way that the right-hand side of the resulting equation is equal to zero. In this way we will show that some combination of the levels is constant. If some combination of levels is constant, we can say that the system is in equilibrium. Thus we seek to characterize the equilibrium state of the generic structure under discussion in a simple way.

In what follows we consider only the question of how to combine equations so that the resultant has a right-hand side equal to zero. This is a simpler problem to solve than to find a transformation which makes the right-hand side equal to zero. To use a transformation, we would have to re-define the level variables and thereby create some artificial, abstract levels contrary to good practice.Furthermore, it is not possible in the general nonlinear case to find such a transformation. This constraint is not so severe for the piecewise linear models commonly found in system dynamics work since they can be treated as a succession of linearized models, with the linearization performed about the trajectory of the full nonlinear system. For this reason, we will deal principally with the linearized version of the generic structure.

Combining the equations shown in Figure 5 is equivalent to creating linear combinations of the row vectors of the system matrix A. The determinant of the system matrix is simply

$$\begin{aligned} \det A &= (-1)^{n+1}\left(\left(\prod_{i=1}^{n} d_i\right)(a'_{21}\cdots a'_{1n}) - 1\right) \\ &= (-1)^{n+1}(A'_{1n}A'_{n,n-1}\cdots A'_{21} - 1) \qquad (45) \\ &= (-1)^{n+1}(\text{OLSG} - 1) \end{aligned}$$

When the determinant is non-zero, the matrix is of full rank and the row vectors are linearly independent; i.e., no linear combination gives a right-hand side which is zero. The determinant is zero only when the open-loop step gain is equal to 1, and this is the only condition under which some linear combination of the row vectors will give a zero right-hand side. Since each such linear combination of the rows reduces the rank of the matrix by one, we can have at most n such linear combinations. We seek all vectors

$$\underline{t}_i = (t_{i1}, t_{i2}, \ldots, t_{in})$$

that are solutions of equations like

$$\underline{t}_i A = (0, \ldots, b_i, \ldots, 0) \tag{46}$$

where b_i is the ith component of the vector. The solution in the general case is given by multiplying on the right by the inverse of A,

$$\underline{t}_i = (0, \ldots, b_i, \ldots, 0)A^{-1} \tag{47}$$

If the right-hand side of equation (46) is zero, then the solution vectors $\underline{t}_i$ are non-zero only when the determinant of A is zero. However, when the determinant of A is zero, the inverse does not exist. Thus we must simultaneously have b_i and det A equal to zero in order to find a non-trivial solution for $\underline{t}_i$. Thus we seek solutions for $\underline{t}_i$ when

$$b_i = \det A \tag{48}$$

and then we see that the $\underline{t}_i$ that we have found remain valid when det A goes to zero. It is clear from equation (47) that the $\underline{t}_i$ are proportional to the row vectors of the inverse of the system matrix. Using this observation, we find the general form of the solution of these linear equations for the components t_{ij} of the vectors $\underline{t}_i$ is [9]

$$t_{ij} = (-1)^{i+j}(\text{cof } A^T_{ij})b_i/(\det A) \tag{49}$$

where $(\text{cof } A^T_{ij})$ is the cofactor of element (i,j) in the matrix A^T, i.e. the determinant of the sub-matrix formed by crossing out row 1 and column j in the transpose of matrix A. It is straightforward to show that the t_{ij} are, except for a factor $(-1)^{n+1}$ which is common to both t_{ij} and det A, as follows:

$$\begin{aligned} t_{ij} &= (A'_{i,i-1}\cdots A'_{j+1,j}),\ j=1,\ldots,i-1 \qquad (50) \\ &= 1,\ j=i \qquad (51) \\ &= (A'_{i,i-1}\cdots A'_{21})\,A'_{1n}(A'_{n,n-1}\cdots A'_{j+1,j}), \qquad (52) \\ &\quad j=i+1,\ldots,n-1 \\ &= (A'_{i,i-1}\cdots A'_{21})A'_{1n},\ j=n \qquad (53) \end{aligned}$$

The solutions for $\underline{t}_i$ show a cyclic appearance of the coefficients A'_{ij}. Note that the only element of A which does not appear in the determination of $\underline{t}_i$ is $A'_{i+1,i}$, the non-unity element in column i of the matrix A or in row i of the matrix A^T (for i=n, the coefficient A'_{1n} is absent). We have found vectors $\underline{t}_i$ such that

$$\underline{t}_i A = (0, \ldots, \det A, \ldots, 0) \tag{54}$$

where det A appears in the ith component of the vector on the right-hand side. When det A is non-zero, each of the $\underline{t}_i$ is independent of the others. We see that the n linear combinations

$$\underline{t}_i\underline{x} = t_{i1}x_1 + t_{i2}x_2 + \ldots + t_{ii}x_i + \ldots t_{in}x_n,\ i=1,2,\ldots,n \tag{55}$$

are constants of the motion when det A equals zero, since

$$(\dot{\underline{t}}_i\underline{x}) = \underline{t}_i A\underline{x} = (\det A)\,x_1,\ i=1,\ldots,n \tag{56}$$

Note that the rate of change of each linear combination in equation (56) depends only on the level x_1 when det A is non-zero. From the previous observation that the vectors

$\underline{t}_i$ are proportional to the row vectors of the inverse of the system matrix, it is clear that the existence of such linear combinations, which are constant when the determinant of A is zero, is not restricted to the generic structure under discussion. We can find such constants of the motion for any linearized system. The unique feature of this structure is that the determinant of A, the eigenvalues, and the polarity of the structure, in essence, all of the fundamental dynamics depend in a simple way on the open-loop step gain.

The necessity of having one minor loop for each level in the generic structure appears again in the observation that if there were one or more integrators, then the limit as the determinant of A goes to zero would imply that at least one of the off-diagonal terms was equal zero, and the major loop would decompose into two or more sub-major loops.

4.1 Example: A Five-level Generic Structure

As an illustration, we consider the case n=5 and show the canonical form and the vectors $\underline{t}_i$. The equations in canonical form are

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 0 & 0 & 0 & A'_{15} \\ A'_{21} & -1 & 0 & 0 & 0 \\ 0 & A'_{32} & 1 & 0 & 0 \\ 0 & 0 & A'_{43} & -1 & 0 \\ 0 & 0 & 0 & A'_{54} & -1 \end{bmatrix} \underline{x} \qquad (57)$$

The vectors $\underline{t}_i$ are

$$\underline{t}_1 = (1, A'_{15}A'_{54}A'_{43}A'_{32}, A'_{15}A'_{54}A'_{43}, A'_{15}A'_{54}, A'_{15}) \qquad (58)$$

$$\underline{t}_2 = (A'_{21}, 1, A'_{21}A'_{15}A'_{54}A'_{43}, A'_{21}A'_{15}A'_{54}, A'_{21}A'_{15}) \qquad (59)$$

$$\underline{t}_3 = (A'_{32}A'_{21}, A'_{32}, 1, A'_{32}A'_{21}A'_{15}A'_{54}, A'_{32}A'_{21}A'_{15}) \qquad (60)$$

$$\underline{t}_4 = (A'_{43}A'_{32}A'_{21}, A'_{43}A'_{32}, A'_{43}, 1, A'_{43}A'_{32}A'_{21}A'_{15}) \qquad (61)$$

$$\underline{t}_5 = (A'_{54}A'_{43}A'_{32}A'_{21}, A'_{54}A'_{43}A'_{32}, A'_{54}A'_{43}, A'_{54}, 1) \qquad (62)$$

4.2 Unity-gain in the n-Level, Linearized Model

When det A equals zero, we see from equation (56) that the rate of change of the n linear combinations of the rows of A vanish. In this case, the coefficients of the matrix A satisfy the relation

$$A'_{1n}A'_{n,n-1}\cdots A'_{21} = 1 \qquad (63)$$

so that one coefficient can be written as the reciprocal of the product of all the others. This is the **unity-gain** case. In fact, the linear combinations are **proportional** to each other with

$$\underline{t}_1 = A'_{21}\underline{t}_2 = \cdots = A'_{n,n-1}A'_{n-1,n-2}\cdots A'_{21}\underline{t}_n \qquad (64)$$

Thus the rank of the matrix formed from the $\underline{t}_i$ as row vectors (and proportional to the inverse of the system matrix) changes from n to 1 as det A goes to zero. Alternatively, one can say that the number of independent constants of the motion changes from n to 1 as det A goes to zero, or as the system goes to a unity-gain condition. In this sense, the unity-gain condition reduces the order of the system from n to 1, further evidence of the singularity of the unity-gain state. This singularity inspired the interpretation of the constant of the motion as the 'circulating content' of the major loop as described in a companion paper[1].

When det A is not equal to zero, the proportionality shown in equation (64) holds for all components of the vectors $\underline{t}_i$ except the ith component. Multiplication by the appropriate product of gain factors shown in equation (64) gives the ith component of $\underline{t}_i$ equal to the gain around the major loop, which is not equal to 1 in the general case so that equation (64) is not valid and the number of independent linear combinations of the levels, or the order of the system, is n.

We have shown that the linearized canonical form can be reduced, on multiplication by certain vectors $\underline{t}_i$, to equation (56) so that the n linear combinations of the states x_i are

- mutually proportional, and
- constant

as long as the system matrix is singular, i.e. as long as det A equals zero. In the case of a general system structure, the constants of the motion are independent linear combinations of the levels with no intuitive interpretations or interrelations. The mutual proportionality of the constants of the motion shown here is a feature of the generic structure under discussion. This feature is simply one aspect of the unity-gain condition, whose study inspired the present work.

4.3 Structural Polarity

We see also that, for the generic structure under discussion here, the dynamic behaviour of the linear combinations of the states is given in an essential way by the sign of the determinant of A. If the sign is positive, the combination grows 'exponentially' since both sides of the equation are linear in the state x_i. Similarly, if the sign is negative, the combination decays 'exponentially'. Recalling that det A is proportional to (OLSG-1) we can see that the growth, decay or constancy of the 'constants of the motion' for this generic structure are determined respectively by

$$\text{OLSG} > 1, \text{OLSG} < 1, \text{OLSG} = 1 \qquad (65)$$

The simple dependence of the dynamics of the structure on the single measure- the open-loop step gain- suggests that we may define the polarity of the structure according to the sign of det A; i.e., that the conditions (65) correspond respectively to

positive, negative, and neutral polarity

of the generic structure. In view of the generality of the result in equation (64), this designation of **structural polarity** may equally well be applied to other structures. However, for general system structures, the polarity will not depend on a simple combination of the system parameters such as the open-loop step gain and may have no intuitively useful meaning.

4.4 Stochastic Effects

System dynamics models are usually developed as deterministic models of organizational and social systems with fixed coefficients and few or no exogenous inputs. Complete testing of a model commonly includes the application of stochastic test inputs to various levels in order to determine the response of the model to random, uncontrolled influences.

With no exogenous inputs, the constant of the motion is strictly constant when the determinant of the system matrix is zero, or the gain around the major loop is unity. Exogenous inputs

$$\underline{u} = (u_1, \ldots, u_i, \ldots u_n)^t \tag{66}$$

to the levels $\underline{x}$ are combined in the same way as the net rates (the derivatives of the levels with respect to time) shown in equation (56) and appear as a second set of terms on the right-hand side of that equation or as shown in equation (67).

$$\underline{t}_i \dot{\underline{x}} = (\det A) x_i + \underline{t}_i \underline{u} \tag{67}$$

When the first term is zero, the exogenous inputs act directly to change the constant of motion. Uncorrelated stochastic inputs are represented by a sequence of pulses of random magnitude. As noted above, when the gain is unity, a pulse input is distributed around the loop so that each level is increased by an amount proportional to the total delay time around the loop. From equation (67), we can infer that the generic structure is more sensitive to exogenous inputs in a period when the constant of motion is passing from a period of growth to a period of decline (or vice versa) so that the first term on the right-hand side is passing from positive to negative (or vice versa). Otherwise, the effects of exogenous inputs are modified by the term proportional to one of the major loop levels. In particular, independent stochastic inputs are more or less correlated in their effect on the constant of motion according to whether the first term is non-zero or zero.

A different kind of problem is posed by the possibility that the coefficients are stochastic. In this case, commonly encountered when considering aggregate relationships between similar elements, the question is whether the stochasticity imparts any significant changes to the average or, more precisely, to the 'certainty equivalent' behaviour. The unity-gain condition is in some respects similar to the 'zero mean growth rate' example proposed by Athans as an illustration of the 'uncertainty threshold principle'[14]. Simply put, if the distribution of the gain about a mean value of 1 is sufficiently wide, the system may show growth rather than the equilibrium condition that would prevail in the absence of the stochastic variation of the gain. Tests of the importance of the uncertainty threshold principle for models with nonlinear feedback control are currently under way.

4.5 Nested Generic Structures and Additive Gains

We noted above that it is not possible to combine two generic structures of the same order and get a structure of the same type. Such a combination includes two independent sets of relationships or flows of casuality linking all of the levels in the structure. The combination of two such structures introduces a welter of cross-links and sub-major loops. It is not surprising that the simple relationships between the gain, the eigenvalues, the constants of motion and the behaviour are not preserved in the combined structure. However, it is possible to generalize the generic structure, to include a certain kind of sub-major loop and the generalization shows some counter-intuitive features. We consider the generic structure in canonical form as shown in Figure 5 with the addition of a single sub-major loop of the same type 'nested' in the larger structure. An example is given in Figure 8 where the sub-major loop linking levels 3 to 5 is included in a major loop linking all 5 levels and only the non-zero

$$\underline{x} = \begin{pmatrix} -1 & & & & A'_{15} \\ A'_{21} & -1 & & & \\ & A'_{32} & -1 & & A'_{35} \\ & & A'_{43} & -1 & \\ & & & A'_{54} & -1 \end{pmatrix} \underline{x}$$

Figure 8: Nested Generic Structures.

elements are shown. Note that the major and sub-major loop structures are both of the same form as the generic structure. It is easy to show that

$$\det A = -1 + A'_{35}A'_{54}A'_{43} + A'_{15}A'_{54}A'_{43}A'_{21} \tag{68}$$

so that

$$\det A = -1 - + \mathrm{OLSG}_{\text{sub-major}} + \mathrm{OLSG}_{\text{major}} \tag{69}$$

where

$$\mathrm{OLSG}_{\text{sub-major}} = A'_{35}A'_{54}A'_{43} \tag{70}$$

= open-loop step gain of sub-major loop

$$\mathrm{OLSG}_{\text{major}} = A'_{15}A'_{54}A'_{43}A'_{32}A'_{21} \tag{71}$$

= open-loop step gain of major loop

By the same procedure as before, we can find vectors $\underline{t}_i$ such that

$$\underline{t}_i A = (0, \ldots, \det A, \ldots, 0) \tag{72}$$

where det A appears in the ith component. In this case it is easy to show that for i=1, for example.

$$t_{11} = 1 - A'_{35}A'_{54}A'_{43} \tag{73}$$

$$t_{12} = A'_{15}A'_{54}A'_{43}A'_{32} \tag{74}$$

$$t_{13} = A'_{15}A'_{54}A'_{43} \tag{75}$$

$$t_{14} = A'_{15}A'_{54} \tag{76}$$

$$t_{15} = A'_{15} \tag{77}$$

When det A is equal to zero, there is a constant of the motion and it is again proportional to $\underline{t}_1\underline{x}$. Note that the only difference in $\underline{t}_1$, in comparison with the example shown earlier, is in the first component which is reduced by the gain of the sub-major loop. The behaviour of the 'constant of the motion' is given by the sign of det A as before. It may happen that

$$\text{OLSG}_{\text{major}} < 1, \text{OLSG}_{\text{sub-major}} < 1 \qquad (78)$$

so that for each piece of the structure, the corresponding constant of the motion that can be defined is decreasing. As noted in Section 4.2, the conditions in equation (78) justify calling the corresponding structure (or sub-structure) a negative feedback structure. However it may also be that

$$\text{OLSG}_{\text{major}} + \text{OLSG}_{\text{sub-major}} \geq 1 \qquad (79)$$

so that two structures that separately are of decay type, may show constancy or even growth of the 'constant of the motion' defined for the whole structure. Again, there is some justification for calling the resulting structure a positive feedback structure since the rates of change of the constants of the motion are positive functions of the levels. The nesting of this kind of generic structure may occur when the value of some level is used simultaneously to form the input to a growth policy as in the term $A'_{15}x_5$, and to form some traditional (smoothed) standard of performance, also in a positve loop, as in the term $A'_{35}x_5$. The possibility of such an occurrence is one aspect of this generic structure to be kept in mind when designing policies or interpreting the dynamic effects of structural changes in a model.

Finally, we note that the previous statement that generic structures are not additive must be interpreted to mean that generic structures whose directions of flow of casuality are different (opposed) are not additive. In this section, the direction of flow of casuality is the same in the major and sub-major loops. The additivity is evidenced by the fact that the dynamics of the whole structure depends essentially on the sum of the gains in the two identifiable generic components.

5. CONCLUSIONS

In this paper we have described some general properties of a generic structure organized around a positive major loop with a full set of minor negative loops. In particular, we have shown how to determine a complete set of independent linear combinations of the levels in the major loop, which are mutually proportional and constant when the open-loop step gain is unity, and which otherwise are sensitive indicators of the underlying dynamics (growth, decay, stabilization) of the system, as a whole. The generic structure discussed here (and its generalization to nested structures of the same type) encourages one to describe the **polarity of the structure** on the basis of the open-loop step gain value being greater or less than 1. When this is the only element being analyzed, we have seen how the eigenvalues, the gain and the 'constants of the motion' are related to the polarity of the structure. However, when other non-generic elements are present, the polarity is not so clearly useful for analytical work.

Under conditions of unity-gain, generic structures of the type we have discussed appear merely to react to external influences, whether they be due to purely exogenous inputs or to the behaviour of non-generic, sub-major loops. It is an open question to what extent this kind of generic structure determines the behaviour of structures external to it. The sensitivity of the constant when the gain is unity argue for expecting structures of this type to exert little influence on the dynamic behaviour of a system. In the context of ongoing work on the characterization of behaviour modes and their sensitivity to parameter changes, [14] it may be possible to use the results developed here to eliminate with some certainty those elements of structure which are completely understood and perhaps unimportant to the fundamental behaviour of interest in order to concentrate effort on understanding those minimal structures needed to explain the behaviour.

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