DYNAMICAL SYSTEMS THEORY AND COMPLICATED ECONOMIC BEHAVIOR

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Abstract

Recent developments in mathematics show that more-or-less random behavior and spontaneously evolving structures can be given analytical and deterministic representations. Both empirical simulation and theoretical models have been developed in economics that have similar capacities. This suggests that we are entering a new period when structural change and inherently unpredictable events can be explained or understood in terms of endogenous economic forces.

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by

Richard H. Day

This paper outlines several important related developments in dynamical systems theory and in mathematical economics. It seems that these developments may provide an improved understanding of how economies work.

I. BASIS FOR A NEW PERSPECTIVE

Classical Mechanics

Analytical dynamics begins with Newton to whom we owe the calculus and the idea of differential equations. All of the problems presented by his theory of celestial motion in its general form have not been completely solved. They challenge mathematicians even now.

Of special interst in the present context is Poincaré's perception that solutions of the N-body problem when $N \geq 3$ can be complicated, far more so than the periodic orbits that had been the focus of attention to that time. In the 1930's, Birkhof demonstrated that there could be periodic orbits of every periodicity, all existing simultaneously. Subsequent investigators established the existence of trajectories that were not attracted even to quasi-stationary orbits but which wandered in a more-or-less random or quasi-random fashion. Since such trajectories are difficult to characterize attention has focussed on their statistical properties, the subject of ergodic theory. A review of the literature and key results will be found in Moser [1973].

Dynamical Systems

Central to the analysis that has emerged from classical mechanics is the view that it is not just the property of particular solutions of differential equations that are of interest. One can never be sure where a given process is at any point in time; its state can never be measured with complete accuracy. Therefore, one wants to study solutions whose initial states are neighbors. This means studying the sensitivity of solutions or their stability.

Moreover, one can't specify all aspects of a real system in one's model equations which may be thought of as subject to perturbations. One wants therefore to study how a model's solutions are influenced by these perturbations. This is the study of structural stability.

The study of the behavior of solutions (1) whose initial conditions are perturbed and (2) whose parameters are perturbed, is, in economics called "comparative dynamics." In the mathematical literature it was given an elegant statement along with a more-or-less comprehensive survey of results (up to that time) in Smale's path breaking and highly sophisticated article on "Differentiable Dynamical Systems" [1967].

An extremely important idea central to the analysis of dynamical systems is the idea of an attractor. Such an object is a set of states in the state space of the system which attracts all trajectories emanating from neighboring points. The attractor does not describe change over time. Indeed, even if it is a regular body behavior on or near it may be complicated. Still it expresses something essential about the long-run behavior of the system.

For many parameter values of a given dynamical system its attractors -- if they exist -- are regular objects described by closed curves or surfaces. For other parameter values they may have an extremely complex structure that cannot be described in any simple way. Such attractors are called strange, a term coined by Ruelle and Takins [1971]. Their existence was already suspected by Poincaré.

Nonperiodic Behavior

In a series of seminal papers Edward Lorenz in the early 60's examined various "forced dissipative systems" approximated by certain quadratic differential equations. His work, which was motivated by an effort to use hydrodynamic theory to explain meteorological variables, suggested the existence of wandering behavior of a highly complex type which he called nonperiodic since it was neither periodic or quasi-periodic. Figure 1 reproduces a projection onto 2-space of a trajectory, simulated by computer, for a fourteen equation model of "vacillation." (Lorenz [1963])

Lorenz's work is noteworthy in both its analysis and in the thoughtfulness with which its author contemplated its implications. It follows a reductionist approach in which a "naturally motivated" or "realistic" model is successively simplified so as to obtain precise results while at the same time retaining salient qualitative features both of the model and of the empirical phenomena to be explained. It

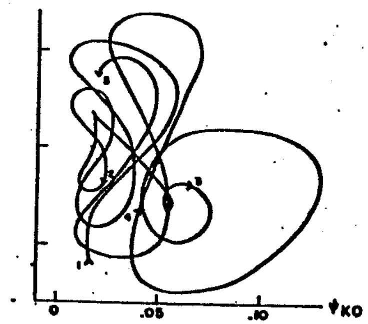


Figure 1: WANDERING TRAJECTORY FOR LORENZ'S VACILLATION EQUATIONS
Source: Lorenz [1963b, p. 459].

also emphasized comparative dynamics in both its senses of stability and structural stability.

Lorenz's work inspired a number of contributions by others and the so called Lorenz attractor has become an intensive object of study by specialists in nonlinear dynamical systems theory. Well known review articles summarize much of this work so I will not survey it further here. But a further aspect of Lorenz's work has a direct bearing on what is to follow and illustrates in the simplest way the basic idea of bifurcation theory.

Bifurcation Theory

Suppose we have the equation

$$(1) \qquad x_{t+1} = mx_t(1 - x_t), \quad x_0 \in [0, 1], \; m \in [0, 4] .$$

(Lorenz used a slightly different but equivalent form.) When $0 \leqq m \leqq 1$ all trajectories converge to $\bar{x} = 0$ which is globally asymptotically stable. When $m > 1$ but ≤ 3 all trajectories move away from 0 and converge to the asymptotically stable stationary state $\bar{x} = 1 - 1/m$. When $m > 3$ but $\leq 1 + \sqrt{6}$, $\bar{x}$ becomes unstable and a stable, two period cycle emerges.

The points $m_1 = 1$, $m_2 = 3$, are bifurcation points where the qualitative behavior implied by (1) changes. They are points of structural instability. As it turns out, $m_4 = 1 + \sqrt{6}$ is also a bifurcation point and in fact as m increases cycles of even, period doubling order emerge. Lorenz also found that as m approaches 4 behavior of such great complexity emerged that no periodic or quasi-periodic motion approximated it; the past did not repeat itself and solutions were highly sensitive to small perturbations in both initial conditions and in the parameter m, exactly the qualitative properties he had found in higher dimension, continuous-time models.

Self-Organization

The capacity of systems to evolve strikingly different qualitative patterns of behavior as a parameter is varied became the basis for Prigogine's celebrated work on bifurcation and self-organization. Imagine a dynamic system characterized by states, some of which move relatively rapidly, described by "variables", and some of which move relatively slowly, described by "parameters." Fixing the slow moving parameters one gets reduced dynamical systems whose variables may converge to some kind of an attractor. Now vary the parameters. The size and shape of the attractor may change gradually as a parameter or several parameters are varied until, when a bifurcation point in the parameter space is passed, the attractor suddenly changes its form altogether. If we imagined that the variables represented particles in space, (atoms, molecules, etc.) what we would observe is the appearance of spontaneous reorganization of the particles, much like a marching band changing formation at the half-time of a football game.

If the parameters ("slow" variables) are subject to random shocks, then, whether or not a system reorganizes in the sense just described is partly a matter of chance. Moreover, once a chance reorganization takes place the change may be irreversible or nearly so. This phenomena is illustrated by the equation

(2) $$\dot{x} = a - bx + cx^2 - dx^3, \qquad a, b, c, d > 0 ,$$

where "a" is a parameter that we may regard as increasing slowly but with small positive or negative random shocks superimposed. See Figure 2.

Suppose the system is initially near state S_1. This state will be observed to move gradually to the right. As time passes it will exhibit small random fluctuations around a gradually increasing state. However, if at any time a positive shock occurs such as to move the system from the general situation in curve 1 to that of curve 2, the system will rapidly evolve to the stable stationary state S_2. Even if secular change in the parameter "a" terminates so that most of the time the phase diagram resumes its initial qualitative form (curve 1) the system, instead of returning to S_1 will remain at S_2.

In a suggestive rendition of language Prigogine refers to this kind of phenomenon as the "self-organization from bifurcation through fluctuations." Obviously, this idea would be of little importance if it were only applicable to situations as simple as that shown in the diagram. But when there are several state variables the change in form can be much more complicated than the jump from one stationary state to another. Then bifurcation can involve altogether different geometric attractors.

Castastrophe

The rapid jump from one kind of motion to another through the bifurcation of parameters or slowly moving variables as exemplified by (1) or (2) have been called "catastrophes," a term coined by René Thom who exploited the idea as a means of characterizing biological morphogenesis; that is, the spontaneous ("endogenous") evolution of new living

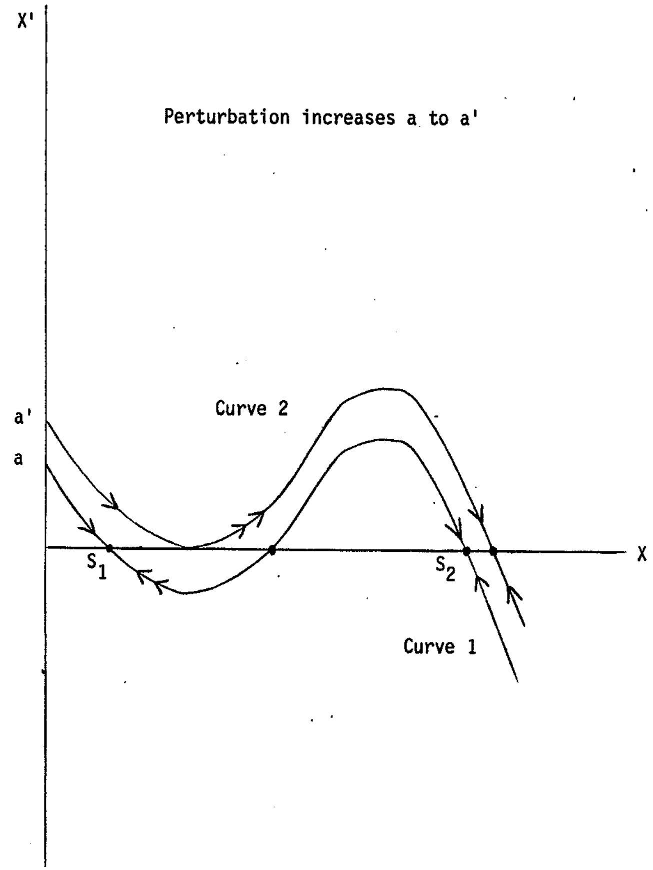


Figure 2: A positive perturbation in "a" sends the system from small fluctuations in the neighborhood of S_1 to small fluctuations in the neighborhood of S_2.

forms from old ones. The idea seems to be essentially the same as that of Prigogine's "self-organization". Thom, however, was able to give an exhaustive characterization of all possible types of catastrophes for dynamical systems with three slowly moving parameters. These have been so frequently described that I will not do so again here.

Chaos

Before turning to economics explicitly, let us linger a bit longer with the kind of nonlinearities underlying all of the complications outlined so far. Lorenz's work established the existence of extremely complex dynamics for a simple difference equation (1). Using techniques developed by Smale, Li and Yorke [1975] discoverd a sufficient nonlinearity or overshoot condition that analytically established the existence of cycles of all orders and a scrambled set in the state space of a single variable system in which all trajectories were non-periodic, and asymptotically unstable, i.e., they moved away from cycles of any order and were sensitive to changes in initial conditions. They referred to trajectories with this character as chaotic, (their paper was entitled "Period 3 Implies Chaos.") The existence theorem was extended to n-dimension discrete dynamical systems ($n \geqq 2$) by Diamond [1976].

The significance of this work lies in its lack of dependence on smoothness of the dynamical system in question. Instead it relies only on continuous and topological properties. This feature has made it possible to develop chaos existence theory for a variety of economic models where non-differentiability is typical.

In Figure 3 the picture shows a continuous but non-smooth, (non-differentiable) mapping on the interval [0, 1]. The points $d \leqq a < b <$

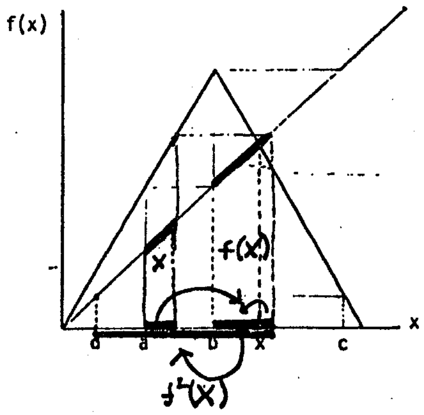


Figure 4: The sufficient overshoot conditions. Points in B map into zero, points in A or A' map into B and so on.

c satisfy the Li-Yorke existence condition and imply the existence of an interval X, say with the properties that

$$(3) \qquad X \cap f(X) = \emptyset \quad \text{and} \quad X \cap f(X) \subset f(f(X)) .$$

One sees how in the process of successsive iterations of the map f (i.e., with the passage of discrete time intervals) states in the set X get mixed and scattered much like the shuffling of cards or molecules of dough in croissants.

Summary

What has emerged from theoretical developments in dynamical systems theory is a gradually improving analytical understanding of complex dynamics. By the latter term I mean (1) unstable nonperiodic (wandering, chaotic, erratic, etc.) behavior and (2) evolving regimes of qualitatively different behavior. Behavior with these characteristics would seem to be nowhere more evident than in economic phenomenon. It may still come as a surprise however, that such possibilities might be generic in the sense that large classes of dynamic processes representing economic behavior can be shown to possess these properties. The next section reviews evidence in support of this assertion.

II. COMPLEX ECONOMIC DYNAMICS

Erratic Fluctuations

Do we need reminding that economic behavior appears to be complicated in the sense just defined? Figure 4 shows some typical macro-

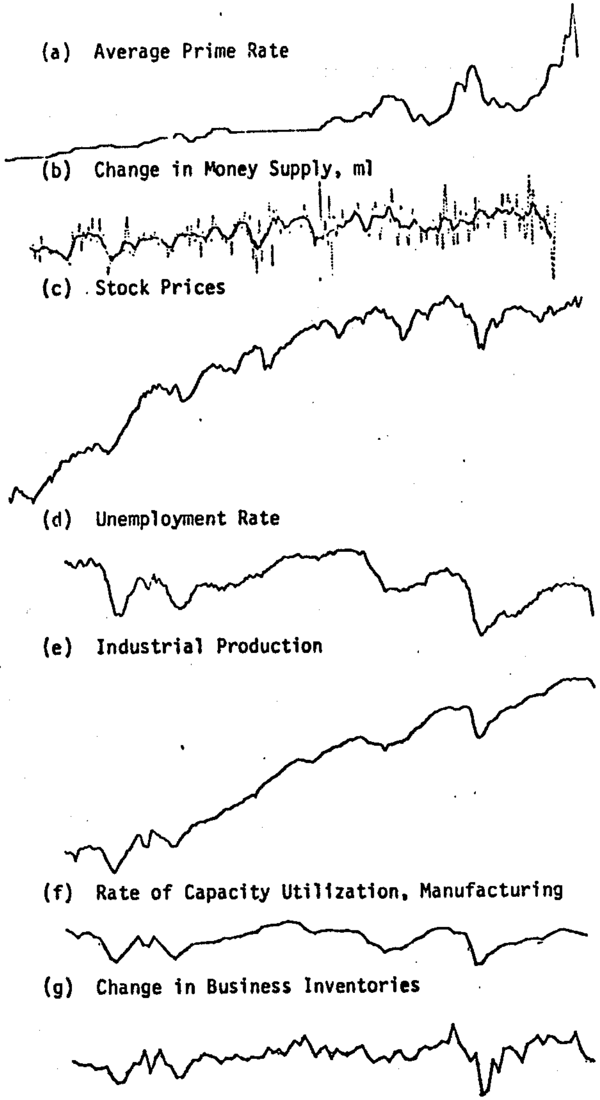


Figure 4: Various Economic Indexes, 1955-1980
Source; Business Conditions Bigest, June 1980

economic series. These data are a little out of date now and more recent figures would add some further drama to the general picture, but perhaps they are sufficient to establish the empirical relevance of wandering, erratic fluctuations in economics.

Traditionally, such patterns have been explained by the superimposition of random shocks on what is (usually) assumed to be a stable, deterministic, linear process. See Sargent [1979, p. 215ff]. That such patterns might be deterministically generated is a novel idea in economics (though one anticipated by Georgescu-Roegen [1954]).

Evolving Regimes

The notion of evolving regimes, that is, successive periods of distinguishably different qualitative behavior, is likewise a commonplace observation to those acquainted with a little economic history or those even modestly acquainted with current events. A few specific examples are always illuminating, however.

Consider energy supplies and prices. After over half a century of nearly exponential growth in energy supplies and monotonic decline in energy costs we have entered a period of fluctuating supplies and prices; after a long period of ranking among the safest of the investment grade corporations, the utilities have become financially insecure, many threatened with bankruptcy. The same may be said for numerous sectors (steel, autos, and so on).

In the realm of individual technologies, we see overlapping waves as a given technique is innovated, then expands first gradually replacing then driving out competitors at an accelerated rate, only to enter an obsolescent phase as another new technique that will eventually replace it altogether enters the picture.

At the level of societies as a whole we see socio-economic ways-of-life gradually grow to prominence, dominate large regions, perhaps rising to such great importance as to be identified as "civilizations," then go into a decline, perhaps precipitous, eventually dying out altogether, some leaving few traces, some leaving vast monuments and widely scattered artifacts to attest to their once grand but mysteriously vanished power.

To many observers the present time seems to be one of rapidly changing futures, trend reversals, newly emerging problems and opportunities, rapidly decaying viability of recently successful economic activity and so on (Forrester [1972].)

"Catastrophe" may be too melodramatic a term to apply to periods of rapid qualitative change, but the existence of such periods can scarcely be argued away. That such periods of transition from one distinct regime to another might be explained by an endogenous theory, while not a novel idea, is at least one that has received little attention within standard or orthodox economics, where, instead, exogenous, ad hoc, explanations are more common. (For example, the innovator or entrepreneur of Schumpeterian theory or the "random shock model" of econometrics.)

Deterministic Dynamic Simulation Models

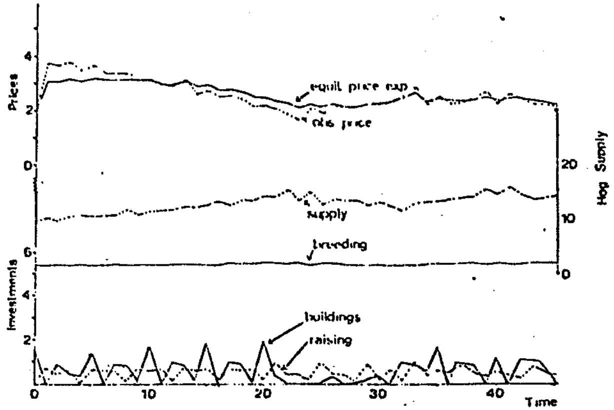
The possibility that patterns of behavior of the kind briefly described could be given an analytical, theoretical representation occurred to me as the result of experience with a class of simulation models designed to simulate production, investment and technological change in various industries and agricultural regions. Early examples of some of this work including studies of the American and Japanese

steel industries, petroleum refining, coal mining and agricultural areas in Brazil, West Germany and the Indian Pinjab will be found in Day and Cigno [1978].

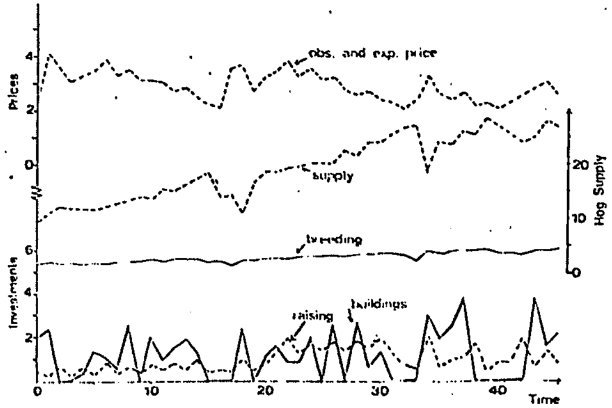
Many of these models are quite detailed involving several commodities, a variety of alternative technologies for producing each one and a considerable number of intermediate products, capital goods and resources. Much in the spirit of Lorenz's reductionist approach, simplified versions of these models were then specified that retained salient features of the large-scale, "realistic" models but which could be analyzed, or failing that, could be studied using computer simulation at low cost.

Three examples will illustrate some typical model behaviors. Figure 5a shows the model generated values for a "corn-hog" model. Illustrated are market and anticipated prices, pork supply and investment in buildings, hog feeding and breeding stocks. Note especially the time profile for investment in buildings. After half a decade a five period cycle seems to appear, being approximately reproduced three times. After period 21 however the time path moves away from this cycle in an irregular oscillation. The effect of an addition of a trend in the demand for pork is shown in Figure 5b. The irregularity of investment is seen again.

These results were obtained from a model specified in the fall of 1968 at Gottingen University by myself, the late T. Heidhues and Garriet Muller now at the University of Frankfurt, using realistic data from West Germany but with the purpose of gaining a better understanding of the rcursive programming modelling approach.



(a) Stationary Demand



(b) Exponential Demand Shifter

Figure 5: SIMULATIONS OF THE GENERALIZED COBWEB MODEL. Müller and Day [1978, pp. 242, 247].

More recently two colleagues (Robert Boyd and Scott Moreland) using a similar type of model produced experimental simulations of the electric power industry that was designed to explain investment in two alternative technologies (capital intensive and capital saving) in the presence of growing demand but under alternative inflationary conditions. We see in Figure 6(a) and (b) two examples. "Cap.I" is capital intensive investment represented by the shaded area. The unshaded area represents investment in the capital saving technique. Total investment is the upper line. The increasing irregularity of both total investment and its composition in response to inflation should be noted. In this model we see the economic effect of a shift in a "parameter" (the inflation rate). It results in extreme shifts to capital saving caused by severe financial strains and changes in the cost of capital.

Here again our effort was directed at representing economizing behavior in a "realistic" way but with little idea in advance as to what might turn up. Yet, the model portrays the shift from a regime of secular expansion and economic health to one of financial crisis and sudden switching of investment strategies in a more-or-less realistic fashion.

Finally, consider Figure 7, which shows the results of another deterministic computer simulation of a recursive programming model of the same general class. The underlying model was designed to represent economic development in a highly populated, open economy initially dominated by agriculture but with an infant industry just beginning to expand. Of special interest is the "overlapping-wave" character of technology shown in Figure 7(a) and the cusp-like switch from growth to decay in fibre and the export crop in Figure 7(b). These drastic

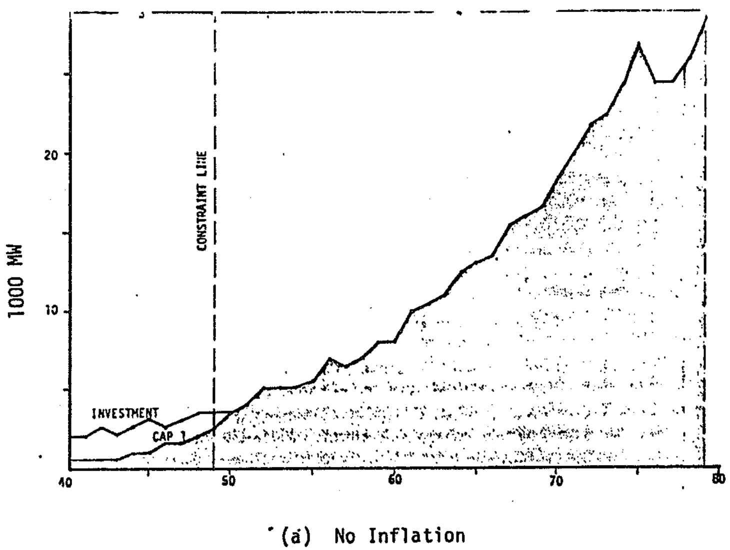


FIGURE 6: SIMULATIONS OF THE EDEM MODEL
Economic Dynamics [1980]

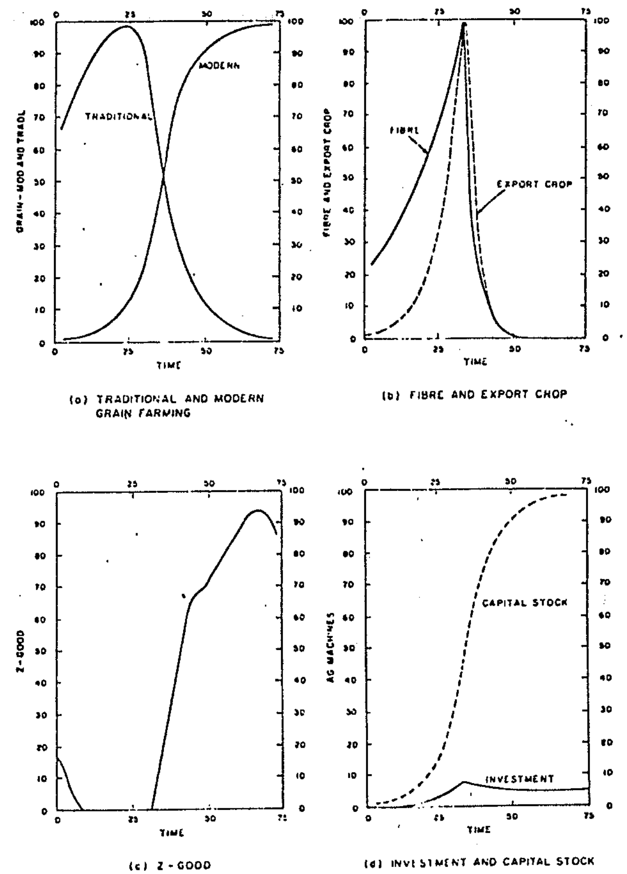


Figure 7: A Dynamic Model of Urban-Regional Development

"structural" changes are occurring while aggregate capital stock accumulates according to the smooth, classic, sigmoid pattern of Figure 7(d). The z-good charted in Figure 7(c) represents the allocation of labor to service activities in the urban sector. Here, as in reality, the advanced stages of growth are accompanied by a vast expansion of the tertiary economy.

Certainly in a conference on System Dynamics one would be remiss in failing to cite simulation models constructed according to the Forresterian paradigm which likewise emphasizes nonlinearity in feedback systems. A search of the diagrams of Forrester's original treatise [1961] or of many of the studies of his followers would reveal numerous additional examples of the phenomenon we are talking about in this paper.

The Robertson-Williams Cobweb Model

The characteristics of the simulation models just reviewed that are responsible for fluctuations in prices, output and investment are (1) the dependence of revenue through feedback on a demand function with variable elasticity, (2) the presence of a financial constraint that depends on revenue, and (3) the independence of pricing from the production and investment decision: either prices or outputs are determined by purely competitive markets. If one begins with sufficiently small initial endowments of working capital there is an inital period of growth. Eventually output levels reach the inelastic portion of demand; consequently, revenues fall. This reduces working capital and borrowing ability for the subsequent period. Production and/or investment must be reduced, or a shift to money-saving production and investment alternatives effected. Later, because market supplies are reduced, demand "recovers," prices increase and output can expand once more. (In the developing context these fluctuations are avoided temporarily by the continued growth in demand.)

D.H. Robertson's observation, made in the context of a discussion of the Keynesian multiplier, that currect expenditures are made from previous income can serve as the basis of a model of extreme simplicity, yet one that can produce many of the features of complex dynamics that we have just been talking about.

Robertson's basic equation

> <u>Current Expenditure</u> depends on <u>Lagged Income</u> becomes, in the context of the firm <u>Current Production Costs</u> are limited by <u>Lagged Revenue</u>,

a statement that reflects John Burr Williams (1967) "current assets mechanism." A sales maximizing hypothesis (Baumol [1959]) coupled with a financial constraint leads to such an equation (Day 1967). Given appropriate aggregation conditions and setting cX equal to total production costs (where "c" is unit cost and X industry output" and D(X, a) equal to demand (where "a" is a parameter measuring the size or "extent of the market" to use Adam Smith's phrase we arrive at the equation

(4) $$X_{t+1} = mX_tD(X_t) \ , \qquad m = a/c > 0 \ ,$$

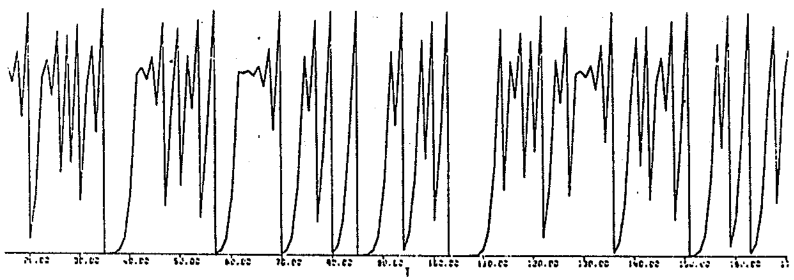
and where for simplicity we have $D(\cdot, a) \equiv aD(\cdot)$. The parameter "m" is therefore a measure of the extent of demand in "efficiency units".

Now it is easy to see that if the revenue function XD(X) has the usual "bell" or "single-humped" shape, m can be increased, revealing a sequence of bifurcation points at which successively higher order cycles emerge accumulating to a critical value, m_c, say, at which the Li-Yorke Theorem is satisfied so that cycles of all orders and unstable chaotic trajectories exist!

Noting that when D(X) = 1 - X (which is equivalent to a - bX under a suitable transformation) we have the classic equation of Lorenz. The simulation of Figure 8 can be thought of as representing a highly volatile industry (hoola hoop or skate boards) that enjoys periods of growth and sporadic booms followed by collapses of greater or less magnitude. Here we have an existence theorem for the irregular fluctuations of an unstable nature displayed in the simulation models.

When I first analyzed this model back in the early 60's, sometime after going to the University of Wisconsin, I was not aware of Lorenz's work on analogous dynamical systems, nor was the Li-Yorke theorem or any of the other work summarized in Part I of this paper available. I was puzzled by the simulations but dropped the bifurcation analysis after obtaining the first several bifurcation points for m Day [1967]. It was only at the suggestion of the mathematician Kenneth Cooke that I looked into Lorenz's papers in 1978. Then in collaboration with Jess Benhabib of USC (now at NYU) who had come across the same literature in connection with quite different work on growth theory, a way was found to use the Li-Yorke theorem to prove existence for a variety of economic models.

The key point in the present model (4) is that the type of nonlinear feedback induced by the financial constraint is going to be generic in economic models that treat financial resources as well as



a) Chaos (1/c = 4.0)

Figure 8

prices. Even assuming linear cost and demand functions a quadratic feedback occurs. The moral is that an economic world in which money enters in a non-trivial way can be highly complex in its behavior in theory just as in reality!

In addition to various papers by Benhabib and myself other studies have begun to appear of economic chaos in a wide variety of settings and I expect this to be one of the most interesting and rewarding areas of theoretical research in economics in the coming few years ahead.

Self-Organization and Catastrophes in Economics

Prigogine's application of bifurcation theory to forced dissipative systems to obtain an explanation of "self-organization" has been given an imaginative application to urban-regional science by Peter Allen et al. (undated) whose results have been obtained using computer simulation. They have shown how the introduction of a single railroad or highway can induce a switch in regime and impell an economy onto a path leading to a new distribution of activity, to bring about self-organization or self-reorganization as it were. Allen (1981) has also contributed a thoughtful exegesis of the general idea as well as its economic application.

As indicated above I think the basic insights of Prigogine and Thom are quite similar, even if different terminology has been used. Certainly they have both stimulated a great deal of imaginative theorizing in a wide variety of fields in social science.

Unfortunately, many of these applications have been "poorly motivated" in the sense that the underlying equations used to explain catastrophes or self-organization in the field in question are not derived

from compelling empirical hypotheses or carefully posed theoretical axioms or propositions. Indeed, criticism by mathematicians has been strong. See for example Sussman and Zahler [1977]. But as with chaos theory, it has proven possible to use catastrophe theory to illuminate dynamic properties of well established theories and models.

In economics "well-established" cannot be taken to mean "accepted" or "non-controversial." Rather I use it to mean that emminent scholars have taken an idea seriously and that it has played an important role in advancing our understanding of important phenomena. Such an example is the paper of Varian [1978] who applied Thom's theory to a clever exegesis of Kaldor's business cycle model, in this way deriving logical consequences of certain behavioral assumptions that were certainly unknown before. Varian forthrightly emphasized the lack of micro-theoretical foundations to the Kaldor macro-model, and this must be candidly admitted as a flaw. Nonetheless, a lack of micro-foundations is not in itself sufficient to justify a charge of "ad hocery", not if the macro-assumptions have sufficient plausibility to invite serious attention, and one does not casually reject an argument set forth by so keen an observer as Kaldor.

Multiple Phase Dynamics in Recursive Programs and Games

The simulation models whose results I briefly described at the beginning of this part belong to a general class of recursive programs and games that explicitly represent economizing but which can incorporate other assumptions of an essentially behavioral nature as well. It is difficult to prove theorems for other than special cases of such models and indeed as far as I know the stationary state and compact orbit theorems of Day and Kennedy [1970] and the Chaos Existence Theorem, Day [1981], are the only "general theorems" that are known.

A central feature of these models is that they incorporate inequality constraints so that at each discrete, decision-making period, economizing can be thought of as selecting a set of positive or active activities and a set of binding constraints. For this reason, although the dynamical system derived for an RP model is set-valued, it possesses solutions that satisfy in a piecewise fashion, sets of difference equations. At each point in time a given set of difference equations prevail but from time to time this set switches so that the solution as a whole can be characterized by an evolving sequence of endogenously generated phase structures, Day [1963], Day and Cigno, [ibid, Chapter 3] and Day [1981].

The result is the appearance of such behaviors as trend reversals, oscillation emerging out of growth and various types of catastrophe or self-organization as illustrated in the examples of Figures 5-7. In the "RP models", however, each phase structure in a given sequence is derived directly from a specific set of economic choices and a specific set of scarce and abundant resources and other limiting factors.

Differential Inclusions

I spoke of the difficulty of mathematically analyzing general recursive programs and games. This seems to be the result of their discrete time, non-differentiable nature. By going to continuous time, adding strong regularity properties and boundary conditions great progress has been made in closely related economic models that lead to differential inclusions where instead of a rate being determined by an

equation, the rate is indeterminant but constrained by a set. Thus, one writes

$$\frac{dx}{dt} \varepsilon A(x)$$

where A is a set-valued map or correspondence.

Most of the work in this area has so far involved monotone solutions of a highly regular type (see Dreze and Valle) and questions of instability, catastrophe, self-organization and evolving phase structures have not yet been posed in this context let alone answered, but a rich analysis has nontheless been developed for such systems leading to the important new book by Aubin and Celina [undated]. Aubin in fact has shown how a decentralized price system obeying a generalized Walras Law can lead an economy to a generally improving performance, while Aubin and Day [1980] have shown how an adaptive economics theory ala Simon, Cyert and March can be formalized using differential inclusions and can be shown to possess improving economic evolutions.

Differentiable Dynamics

Although economics naturally leads to non-differentiable, set-valued dynamical processes, much progress in dynamical systems theory has exploited differentiable, single-valued systems. This includes virtually all of the work mentioned in section 2 except the Li-Yorke existence theorem. In order to make progress in analytical understanding of complex economic dynamics it would seem worth giving up some of the realism of discrete time, nondifferentiable, set-valued structures and to study instead dynamic economic models without these complications but that retain the essential nonlinearities that are there in reality and that lead to complicated behavior of a qualitatively realistic type. This would make many of the techniques developed for nonlinear systems applicable to economics and might accelerate progress just as the calculus helped establish neoclassical theory on a rigorous footing before modern convex analysis and topological techniques were innovated.

EPILOG

Recent developments in mathematics show that more or less random behavior and spontaneously evolving structures can be given analytical and deterministic representations. When applied to a specific field of scientific inquiry they provide possibilities for endogenous theories of complicated dynamics, that is, theories which explain irregular fluctuations and evolving structures by underlying material, mental and social forces rather than by "random shocks," "great men", "the weather" or other unexplained outside or "exogenous" events.

Nowhere do we observe complicated behavior more frequently than in economics. We would therefore seem to be standing at a threshold across which lies a new intellectual domain in which events may be recognized as more or less unpredictable, but nonetheless understandable. What this may mean both for the further progress of scientific method and for practical policy one can only guess.

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