# MODELING STOCHASTIC PROCESSES WITH SYSTEM DYNAMICS

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ABSTRACT

System dynamics as a methodology has traditionally been concerned with the study of processes that can be descibed by continuous variables. Discrete or integer events, such as the number of sales made in a day or the number of factory closings in a year have either been approximated as continuous variables or else not dealt with. This paper examines another way of dealing with discrete events through the realization that any discrete event has a certain probability of occurance. These probabilities are continuous and conserved quantities and can be modeled as system dynamics levels.

Treating probabilities as levels in dynamic simulations is a standard technique in stochastic modeling, markov models being one example. System dynamics' advantage over these other methods is that it can represent the impact of the results of the probabilistic study on social feedback systems.

This paper focuses on examples demonstrating the use of system dynamics to model uncertain events. These examples deal with the simple case of a Poisson process with a time varying event arrival rate. Extensions incorporating conditional and independent probabilities are also considered.

## 1. INTRODUCTION

There are many events and system states that cannot be modeled with traditional system dynamics because they cannot be described by continuous variables. Whether or not a machine is working, and the number of house sales made by a real estate agent in a given day must by modeled as integer, or discrete events. This paper shows how discrete events can be modeled in a system dynamics environment. Though the events themselves are discrete, the probability that these events will occur is a continuous variable.

It is possible to enumerate all possible states of the discrete system under consideration (a machine is either broken or working; real estate agents can sell either no houses, one house, two houses, etc.). If all possible system states are considered, then the system can always be described by one of these states. That is, the sum of the individual probabilities that the system is in one of the considered states will be one at all points in time. Because the total probability is always one, probability is a conserved variable. Because probabilities can take on any value between zero and one, probability is also a continuous variable.

As a conserved and continuous variable, probability can be modeled as a level within a system dynamics model. A set of five interconnected levels can be used to represent the probabilities that a system under consideration is in one of five possible states. As time progresses conserved flows between these levels can deal with the changing probabilities for the system states.

## 2. MOTIVATION FOR THIS STUDY

Models of stochastic processes have existed for a number of years without system dynamics; continuous markov models being one representitive example. The question which naturally arises is whether or not there is any advantage to employing system dynamics instead of the more conventional methods of dealing with



probabilistic systems. If the analysis focussed solely on a description of the discete events then there is probably no reason to favor system dynamics over traditional methodologies. However, many problems involve other factors that influence, and are influenced by discrete events. The need to represent these factors is the driving force behind the choice of system dynamics as the appropriate modeling tool.

Consider the case of a salesman trying to develop a new customer. The salesman must invest some time and energy in developing the human side of the relationship with the potential customers purchasing agents. Modeling the process of building up such relationships is appropriate for classic system dynamics. The relationship can be modeled with a level. The dynamics of increasing this level are understood by salesman and can be incorporated into the model. However, converting the human relationship into a sale is an uncertain event. The level of the human relationship will influence the probability of a sale but the purchasing agent may not have an application for the product, or he may be waiting for a current supplier to make a mistake. By modeling sales as chance occurances we can find sales strategies that maximize expected sales or which minimize the likelihood of achieving no sales.

This paper presents some simple examples where probabilities are treated as system dynamics levels. A first example considers the possibility of failure of a single machine. The results are compared to those using standard stochastic models. Later examples build on this first one to show representations of more complex systems. These examples show how system dynamics can represent time varying expected failure rates as well as conditional and independent probabilities. Finally the system dynamics approach to probability is related to advanced stochastic analysis techniques, such as markov models.

We can define a set of possible states for a probabilistic system. This set is said to be collectively exhaustive if the system is always in one state on another. If in addition, only one state definition applies to the system at any one time, the state set is mutually exclusive. A system will always be in one, and only one, member of a mutually exlusive, collectively exhaustive set. If we are uncertain as to which state the system is in we can say the that the probability it is in one of the states sums to one over such a set.

## 3. A SIMPLE EXAMPLE

Consider a water pump. The pump is either working, or it is not. These two alternatives provide a mutually exclusive, collectively exhaustive set of system states. Suppose the pump is currently working, and it is known that working pumps have an expected failure rate is L failures per year (or alternately, that a working pump has an expected service life of 1/L years). The probability that the pump will fail over a very short time interval, dt, is L*dt. If there is only a finite probability that the pump is working then the expected number of pump failures will be



$$\text{Expected failure rate} = L * \Pr[\text{pump working}] \qquad \text{(Eq. 3-1)}$$

We can rephrase the problem by considering the probabilies as system dynamic levels. The probability that the pump is working is a level which will vary from zero to one. Similarly, a second level will contain the probability that the pump has failed. The rate at which "probability" moves between these levels is the expected failure rate (the probability of pump failure over a short dt is the product of expected failure rate and dt). A flow diagram of the process is presented in Fig. 1.

For the simple case of constant mean failure rate, L, we have defined a classical Poisson process. The probability of pump failure will exhibit the growing exponential expected by students of both probability and system dynamics.

$$\Pr[\text{pump failure before time } T] = 1 - e^{-LT} \qquad \text{(Eq. 3-2)}$$

## 4. A MORE COMPLEX EXAMPLE

When the defining parameter L is constant there is no need to go beyond the simple exponential formulation presented above. However, reality exhibits substantial non-linearities. For the case of a water pump we might expect that rates of failure would increase with service life. Fig. 2 presents a possible relationship between service life and mean pump failure rates. As the pump ages, its expected failure rate increases. Another way of

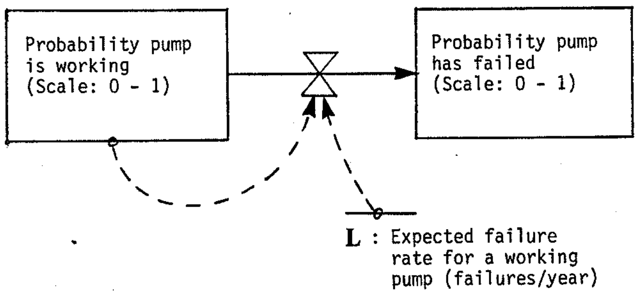


Figure 1: Flow diagram for a simple Poisson process

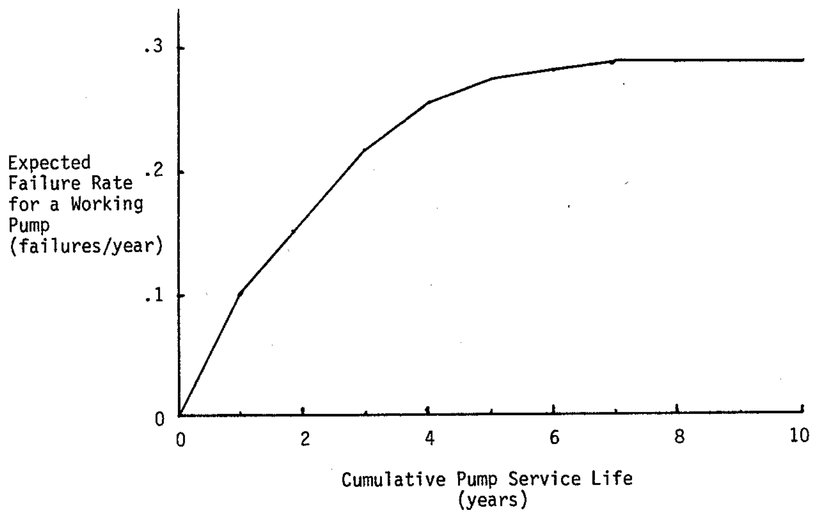


Figure 2: Relation between pump service life and expected failure rate



looking at the table in Fig. 2 is to consider the failure rate as the reciprocal of expected remaining service life. In Fig. 2 we are assuming that a pump that is still operational after one year can be expected to last, on average, ten more years. If it manages to survive up to ten years, then it can be expected to last for a little over three more years.

This relation was incorporated into a DYNAMO model which plots the cumulative probability that the pump has stopped working over time, given that it was originally operational (Fig. 3). For comparison, Fig. 3 also presents the case of simple exponential growth shown in Equation 3.2.

5. CONDITIONAL PROBABILITIES

Suppose our pump from the last example has managed to say in service for 10 years. We decide to replace it with a new pump. However, the old one is still functional so we decide to leave it installed, using it only if the newer pump breaks. The new system has three possible states: (1) the new pump is working; (2) the new pump has broken, but the back-up is functioning; and (3) the entire system has failed. The system is diagrammed in Fig. 4. The system cascaded probabilities between the three levels because failure of the back-up pump is conditional on the prior failure of the main pump.

Note that the expected failure rate differs for the two pumps. The new pump has the same time/failure rate trade-off

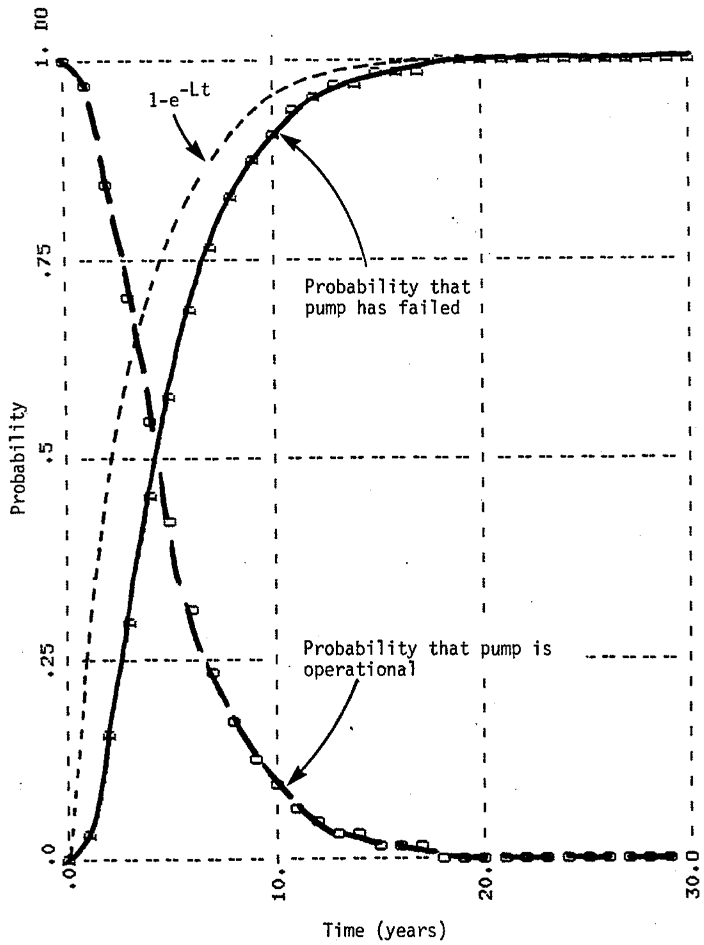


Figure 3: Cumulative probability of failure, where expected failure rate varies with service life



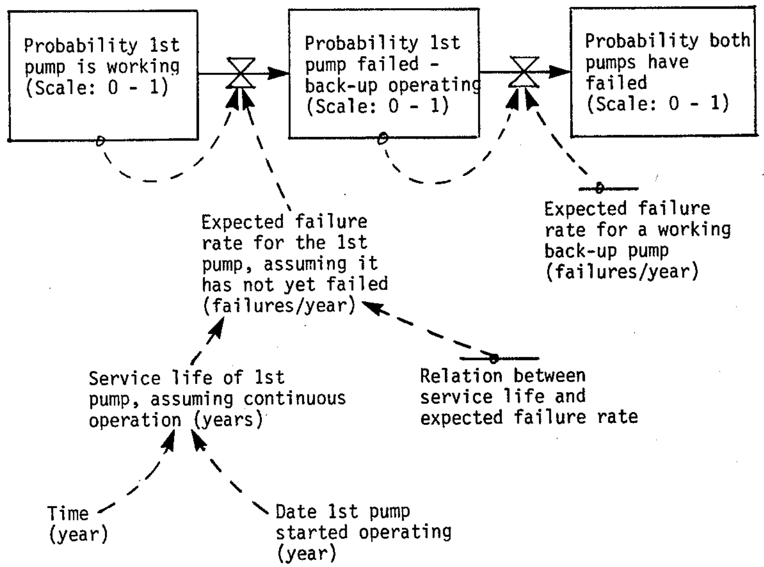


Figure 4: Flow diagram of pump system showing cascading conditional probabilities

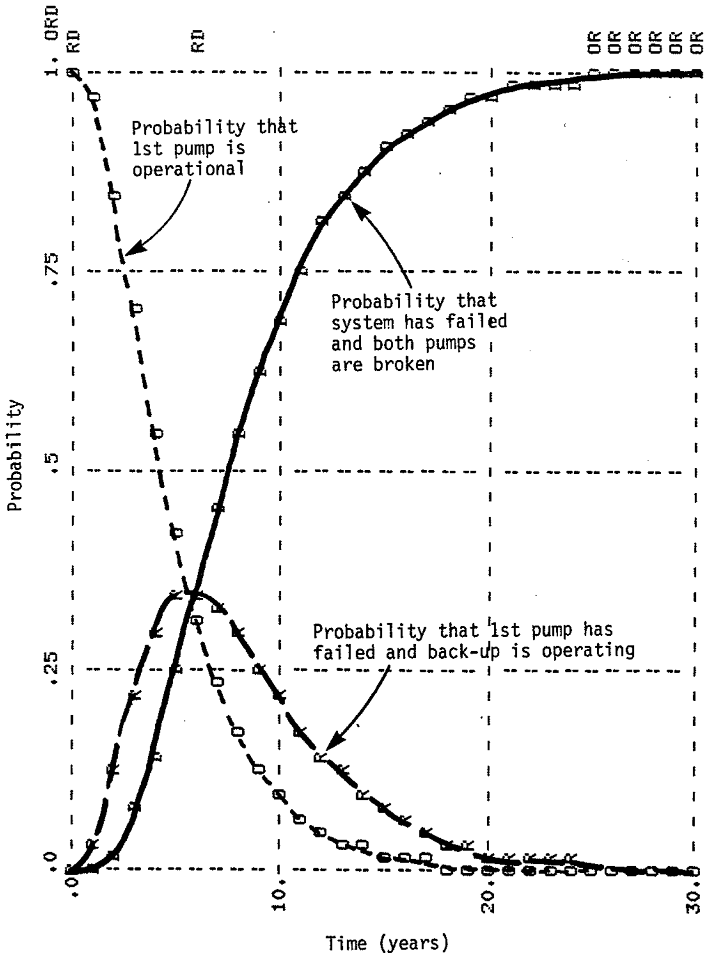


Figure 5: Probability of failure in redundant system



assumed in the first example while the old pump has saturated the curve. The results of this simulation are presented in Fig. 5. The non-linearities in such a probabilistic system are easily handled using system dynamics.

## 6. INDEPENDENT PROBABILISTIC EVENTS

Consider the case of independent pumping systems. If each pump has a 100 gallon/day capacity, then a parallel arrangement will be able to pump 200 gallons/day. Suppose we install a second pump along side the double pump system from the last example. Fig. 6 shows a flow diagram representing this situation.

Because the two pumps are independent, "probability" does not flow between the sub-systems. Within each sub-system, however, probability is conserved and sums to one unit. Using such an arrangement we can determine the probability that the two pump system will supply 200 gallons/day, 100 gallons/day (for the case of either pump broken), or nothing. Fig. 7 shows the relevant DYNAMO output.

## 7. CONCLUSIONS

The above examples could all have been handled by traditional stochastic modeling, such as markov analysis. DYNAMO's simple integration algorithm has perhaps limited the accuracy of

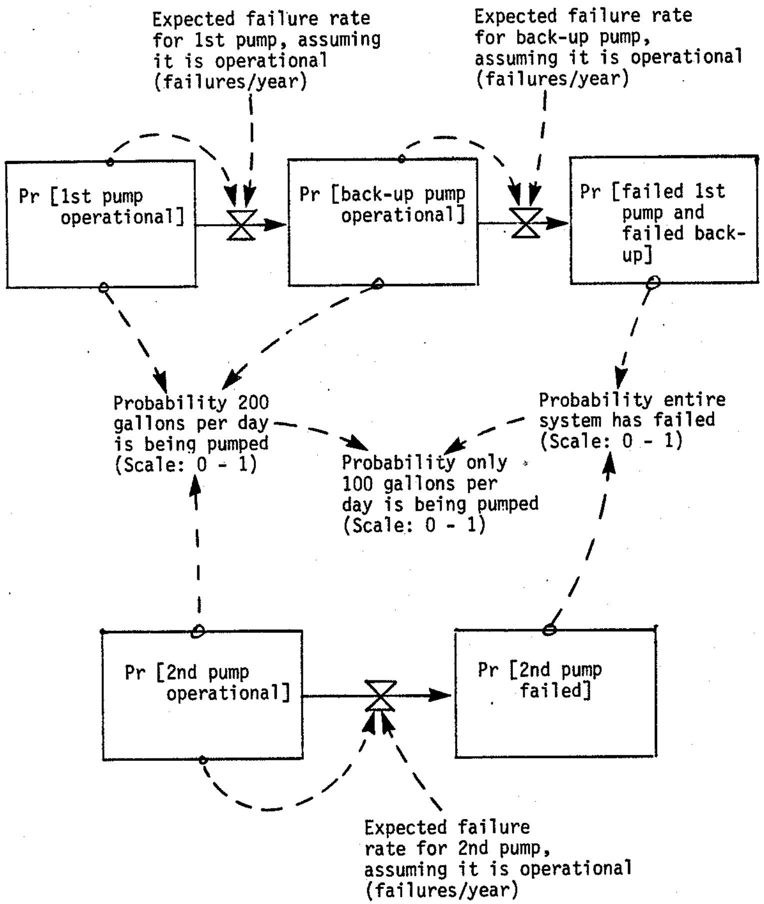


Figure 6: Flow diagram of water pumping system showing independent probabilistic events



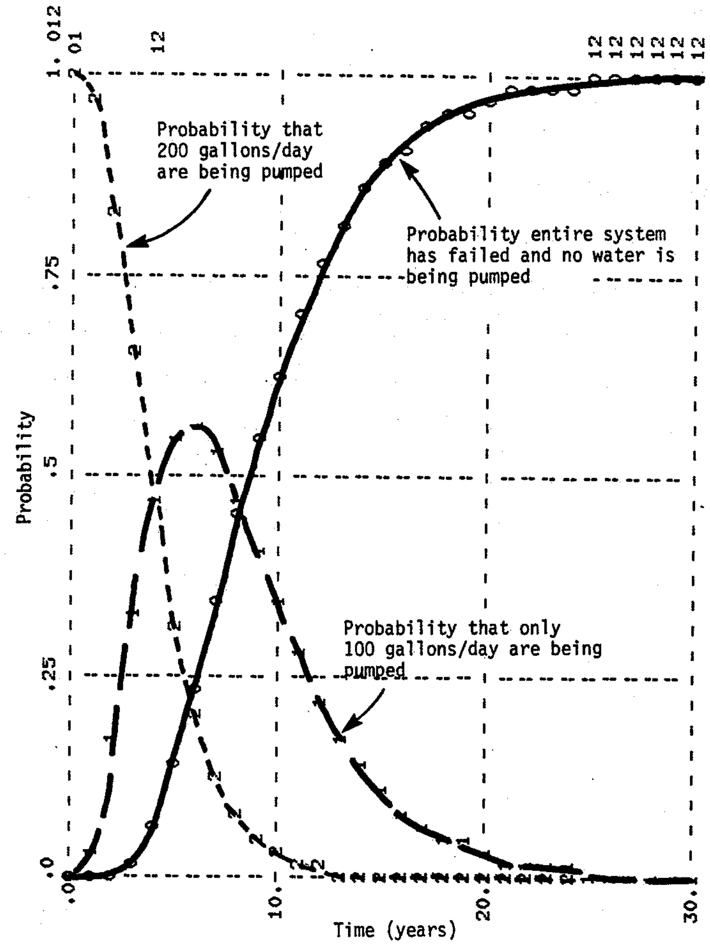


Figure 7: Probability that system is delivering various volumes of water

the results; however DYNAMO's well-defined structure makes the process of setting up the models, especially the ancilliary equations relating failure rates to service life, very smooth. Both system dynamics and markov models involve the enumeration of system states and dynamic simulation of changing state probabilities, and if the analysis were limited to the probabilistic events, there is not much to favor one methodology over the other.

In many situations, though, the concern is with the implications of the state probabilities, rather than with the probabilities themselves. To add a story to the example from Figures 6 and 7 consider an isolated town which requires a water supply of 200 gallons per day. The health of the population and the success of local agriculture will all be affected if pump failures occur. The feedback between health and productivity, and agricultural output and health are processes that can effectively be modeled with the system dynamics methodology. Traditional stochastic modeling is not as adept at dealing with the impacts of probabilistic events on social feedback systems as is necessary for many studies.

On the other side of the coin, it is important for system dynamics practitioners to remember that most systems are not deterministic and do contain some random elements. Monte carlo simulations using some "noise" function is one approach for dealing with uncertainty. However, this technique is not very effective for studying low probability events because of the large cost involved in running many repeated simulations. Model-



ing probabilities explicitly as levels is just another way of dealing with uncertainty in systems.