BIFURCATIONS AND CHAOTIC BEHAVIOUR IN A SIMPLE MODEL OF THE ECONOMIC LONG WAVE

by

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ABSTRACT

This paper presents a total stability analysis of a simplified Kondratieff-wave model. The purpose is to show how such an analysis can be carried out and to illustrate the kind of information one obtains.

For normal parameter values the Kondratieff wave model has a single unstable equilibrium point. Combined with non-linear constraints in the model's table-functions, this instability creates a characteristic limit cycle behaviour. For other parameter values, however, the model is stable and generates damped oscillations instead of the limit cycle. For yet other combinations of parameters, the non-linear constraints yield to the instability, and sustained exponential growth or total collapse result.

By means of linear stability analysis we first determine the conditions for the transition between a stable and an unstable equilibrium to take place. This transition is known as a Hopf-bifurcation. Using global analysis we outline the phase-portrait of a fully developed limit cycle. By the same method, we examine the conditions under which the non-linear functions fail to contain the system so that exponential run-away or collapse occur. A DYNAMO-program is then developed which calculates the Lyapunov exponents of the system during a simulation, and we discuss how these exponents can be used as a measure of the divergence or convergence of nearby trajectories. Finally, we illustrate how subsequent period doublings and chaotic behaviour can occur if the model is driven exogenously by a weak sine-wave, representing for instance the shcrt term business cycle.

INTRODUCTION

During the last few years we have witnessed the development of two different schools within the System Dynamics community. One school, centered in North America, emphasizes its concern with real world problems such as national economic development, resource management, and top-level decision making. The other school, influenced by Prigogine, Allen and their co-workers in Brussels, is engaged in mathematical analyses of bifurcations, chaos and other forms of instability that occur even in relatively simple non-linear models.

The views that these two schools have developed on several basic issues appear quite conflicting. While the North American

school hypothesis that social systems generally are stable and insensitive to parameter variations, the Brussels school postulates all social (and biological) systems to be inherently unstable, and particularly emphasizes situations in which small parameter shifts change the qualitative behaviour of a system. While the Forrester school tends to consider chaos as a phenomenon mainly of academic interest, the Prigogine school postulates chaos to be the usual mode of behaviour for constraint (non-linear) systems with more than two state variables.

In the present paper we shall contribute towards the unification of the two schools by analysing a typical problem from the Forrester school with the mathematical tools used by the Prigogine school. From a basic scientific point of view, we do not think that one can avoid accepting the relevans of chaos and the ideas of Prigogine and his co-workers. But equally important is the (generally unrecognized) fact that the System Dynamics method developed by Forrester is complete and self-consistent enough to incorporate these ideas without any change. We believe that the two approaches should be used to complement one another: clasical Systems Dynamics should recognize the relevance of unstable behaviours in the same way System Dynamics has already added the dimension of dynamical behaviour to the analysis of social systems; the Brussels school should emphasize the development of more realistic models when dealing with social and economic issues.

THE MODEL

Figure 1 shows a System Dynamics flow-diagram for a simplified version of the economic long-wave model developed by

Sterman[1]. This model explains the Kondratieff-wave in terms of subsequent expansions and contractions of the capital-goods sector of an industrialized economy as it adjusts to the required production capacity. The existence of "capital self-ordering", that is the positive feedback loop associated with the fact that the capital sector depends on its own output for an expansion of its production capacity causes the model to be unstable. Even if the model is initiated in its equilibrium point, the slightest disturbance triggers an expanding oscillation.

Figure 1

Well away from the equilibrium point, the model behaviour is confined by non-linearities. One such non-linearity arises from an assumed saturation of the capital order rate as desired production exceeds potential output by more than a factor of 2. Another non-linearity is associated with the upper limit to the utilization of a given production capital. For normal parameter values, the model therefore exhibits a characteristic limit cycle behaviour. In accordance with the statistical results obtained from economic time series analyses by Kondratieff[2] and others[3], the period of the cycle is typically 50-60 years.

In the real world, several other positive feed-back loops are at work, just as many other non-linear mechanisms help to contain the swings of our economy. Some of these mechanisms are discussed by Sterman[4], and are also involved in the System Dynamics National Economic Model[5,6]. These mechanisms will not be considered here. The simple one-sector model has purposely been

designed to isolate and study one particular mechanism which appears important for the development of the economic long wave. Until the dynamics of this model are fully understood, we would be reluctant to proceed to more complicated models.

Our simplified Kondratieff-wave model contains two state variables: the in-place production capital PC, and the backlog of unfilled orders for capital UOC. Production capital divided by the capital/output-ratio COR (assumed constant) gives the potential output PO, and together with the capital utilization factor CUF, this determines the actual production rate PR. The capital utilization factor is a function of desired production DP relative to potential output PO. The non-linear relation assumed for CUF is depicted in the lower right corner of figure 1. The function is defined such that capital utilization is unity when desired production equals potential output. As the desired production exceeds the potential output it is assumed that capital utilization can be increased slightly, but even when DP>>PO, CUF never exceeds a certain maximum level.

As further simplifications we consider the desired production of goods DPG to be constant, and we assume that production of goods has a higher priority than capital production. By subtracting DPG from the total production rate PR we then obtain the rate of capital production which is taken to equal the capital acquisition rate CAR. Capital depreciation is exponential with an average life-time of capital ALC.

Acquisition of new capital automatically reduces the backlog of unfilled orders for capital. New orders are assumed to be placed at a rate OR which equals replacements of discards modified by the multiplier from desired production MDP. The function that we have used for MDP is also sketched in figure 1 (lower left corner). The multiplier is defined such that the capital order rate equals capital depreciation when desired production corresponds to potential output. As desired production exceeds potential output, MDP increases rapidly to saturate at a value of 4.0, when DP/PO>2. To close the loop, desired production DP is finally determined as desired production of goods DPG plus the desired production of capital, the latter being defined as unfilled orders for capital divided by a normal delivery delay for capital DDC.

The simplified economic long-wave model is dominated by the so-called self-ordering loop[1]. This is a positive feed-back loop which in many respects is similar to the accelerator loop of ordinary economic accelerator-multiplier models[7]. In the present model this loop connects capital order rate to desired production, desired production to unfilled orders for capital, and unfilled orders for capital back to the capital order rate. The complete DYNAMO-program for the model is given in App. A. As base case parameter values we have taken:

capital/output-ratio	COR = 6 years
average capital lifetime	ALC = 20 years
delivery delay for capital	DDC = 3 years

The purpose of our analysis is to examine changes in the characteristic mode of behaviour as these and other parameters change. The base case parameters are not realistic of actual economies. In particular the capital/output ratio COR is far too large. However, the purpose of the analysis is not to explain the origin of the long wave but to illustrate the use of various analytical techniques. Though the model is based on the simple Kondratieff-

wave model developed by Sterman[1], it has been simplified so as to make the analysis more transparent. The full simple model generates the long wave with more realistic parameters.

Figure 2a shows a simulation performed with the above parameter values. Through an initial transient, the model approaches a limit cycle with a period of approximately 80 years. The characteristic features of this type of behaviour may be more evident from the phase-plot in figure 2b. Here we have plotted simultaneous values of production capital and unfilled orders for capital (both normalized relative to the desired production of goods) during the evolution of the system. Note how the trajectories approach the limit cycle independent of the initial conditions. If the model is started at a point inside the limit cycle, the oscillations grow, and vice versa. The limit cycle is said to be an attractor for the system.

Figure 2

Our model is in equilibrium when the three rates OR, CAR, and DR are equal. It is a simple matter to show that there exists only a single equilibrium point which is determined by

$$\frac{PC}{DPG} = \frac{COR \cdot ALC}{ALC - COR}$$

and

$$\frac{UOC}{DPG} = \frac{COR \cdot DDC}{ALC - COR}$$

The position of this point is also indicated in figure 2b. For the base case parameter values the equilibrium point is unstable. This means that even if the model is started close to equilibrium, the trajectory will move away from this point and approach the limit cycle. Indeed, even if the model is initiated in equilibrium, computational noise amplified by the positive self-ordering loop is enough to start the limit cycle. On the other hand, if the capital/output-ratio is reduced from its base case value of COR = 6 to COR = 4, the equilibrium point becomes stable, and the model exhibits damped oscillations. This is illustrated in the time- and phase-plots of figure 3. Now the trajectories approach the equilibrium point regardless of the starting point in phase-space.

Figure 3

To the extent the model represents the interactions which produce the economic long wave , the stability properties of the model correspond to those of the real world. Thus, if by some means our economy was brought into equilibrium, the consumption on a hot summer day of one single bottle of beer more than produced could trigger the Kondratieff wave. It might take a few hundred years for the wave to fully develop, but once distorted the economic system would never be able to restore an unstable equilibrium

condition. The build up of coherent macroscopic oscillations by amplification of random microscopic disturbances is a somewhat simplified example of what the Prigogine school calls a self-organizing process[8]. Such behaviours have long been generated and explained in classical System Dynamics models such as the commodity model (Meadows) and the Lotka-Volterra predator pray model (Nathan Forrester).

LINEAR STABILITY ANALYSIS

The simple Kondratieff-wave model can be recast into two non-linear coupled first-order differential equations (one for each state variable):

$$\frac{dPC}{dt} = \frac{PC}{COR}\, CUF\{DP/PO\} - DPG - \frac{PC}{ALC} \qquad (1a)$$

and

$$\frac{dUOC}{dt} = \frac{PC}{ALC}\, MDP\{DP/PO\} - \frac{PC}{COR}\, CUF\{DP/PO\} + DPG \qquad (2a)$$

Here, CUF{DP/PO} and MDP{DP/PO} represent the two table-functions for which the independent variable can be expressed as

$$\frac{DP}{PO} = \frac{UOC/DDC + DPG}{PC/COR}\,. \qquad (3a)$$

To simplify the mathematical treatment let us substitute

$$\frac{PC}{DPG} \rightarrow x, \quad \frac{UOC}{DPG} \rightarrow y, \quad \frac{DP}{PO} \rightarrow z$$

$$COR \rightarrow c, \quad ALC \rightarrow \tau, \quad DDC \rightarrow p$$

$$CUF\{\} \rightarrow f(z), \text{ and } MDP\{\} \rightarrow g(z).$$

The governing equations of motion can then be rewritten as

$$\dot{x} = \frac{x}{c}f(z) - 1 - \frac{x}{\tau} = P(x,y) \qquad (1b)$$

and

$$\dot{y} = \frac{x}{\tau}g(z) - \frac{x}{c}f(z) + 1 = Q(x,y) \qquad (2b)$$

with

$$z = \frac{y/p + 1}{x/c} \quad . \qquad (3b)$$

To determine the stability properties of the equilibrium point, we only need to know the form of the differential equations in the neighbourhood of this point. The model is constructed so that in equilibrium DP = PO (desired production equals potential output), and consequently $z = 1$. Close to the equilibrium point, the two table-functions can therefore be approximated by

$$f(z) \simeq 1 + \alpha(z-1) \quad \text{with } \alpha = \left.\frac{df}{dz}\right|_{z=1}$$

and

$$g(z) \simeq 1 + \beta(z-1) \quad \text{with } \beta = \left.\frac{dg}{dz}\right|_{z=1}$$

Figure 4 illustrates how α and β are defined as the slopes of the table-functions in $z=1$. Typical values of the two parameters are $\alpha = 0.75$ and $\beta = 3$.

Figure 4

We may now proceed to solve the coupled linearized equations

of motion in terms of two linearly independent eigenfunctions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. If the real part of the complex conjugate eigenvalues is positive, $Re\{\lambda_1\} = Re\{\lambda_2\} > 0$, the solutions are exponentially growing, and the system is unstable.

A possible procedure is to determine λ_1 and λ_2 as the eigenvalues of the Jacobian (or functional) matrix [9]

$$\underline{\underline{J}} = \begin{Bmatrix} P'_x & P'_y \\ Q'_x & Q'_y \end{Bmatrix}$$

with P(x,y) and Q(x,y) as defined by (1b) and (2b) evaluated at the equilibrium point. This gives

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \frac{1}{2}(P'_x+Q'_y) \pm \frac{1}{2}\sqrt{(P'_x+Q'_y)^2 - 4(P'_xQ'_y-P'_yQ'_x)} \qquad (4)$$

with the partial differential coefficients

$$P'_x = \frac{\partial P}{\partial x} = \frac{1}{c}(1-\alpha) - \frac{1}{\tau}$$

$$P'_y = \frac{\partial P}{\partial y} = \frac{\alpha}{p}$$

$$Q'_x = \frac{\partial Q}{\partial x} = -\frac{1}{\tau}(\beta-1) - \frac{1}{c}(1-\alpha)$$

$$Q'_y = \frac{\partial Q}{\partial y} = \frac{\beta c}{\tau p} - \frac{\alpha}{p}.$$

Since the average lifetime of capital for any realistic system must be larger than the capital/output-ratio ($\tau > c$), we have

$$P'_xQ'_y - P'_yQ'_x > 0.$$

The last term in the square root of eq. 4 is therefore always negative, and the condition for neutral stability, $Re\{\lambda\} = 0$, reduces to

$$P'_x + Q'_y = 0.$$

or

$$\beta = \frac{ALC \cdot DDC}{COR}\left[\frac{1}{ALC} + \frac{\alpha}{DDC} - \frac{(1-\alpha)}{COR}\right].$$

Figure 5

Figure 5a shows a set of neutral stability curves plotted with the capital/output-ratio COR as the independent variable and the slope α of the capital utilization function as a parameter. The curves were calculated for an average capital lifetime of ALC = 20 years and a normal delivery delay of DDC = 3 years. The neutral stability curves can be interpreted as follows: for given values of α, β, and COR, if the point (COR, β) falls below the α-curve, the equilibrium point is stable, otherwise it is unstable.

As an example, the point COR = 6, β = 3.0 is found to lay above the neutral stability curve corresponding to α = .75. The Kondratieff-wave model is therefore unstable for this set of parameter values, as verified by the simulation in Fig. 2. It is interesting to note that for given values of β and α, the Kondratieff-wave model can be unstable both for high and for low values of the capital/output-ratio with a stable region in between. This is a rather unexpected result which could certainly cause confusion if instead of a formal analysis, the stability properties of the model were investigated only by simulating with many different combinations of parameter values.

With at least 5 different parameters in the problem, there are many ways to present the neutral stability curves. Figure 5b shows the stability curves as function of COR for various values of the delivery delay for capital. α = 0.75 and ALC = 20 years are here taken to be constant. All the curves now cross at a point corresponding approximately to COR = 5 years. At this point the stability properties of the model are completely independent of the delivery delay for capital. For higher values of the capital/output-ratio, an increase in the delivery delay for capital reduces the stability of the model, while for smaller values of COR (COR<5) increasing the delivery delay gives a more stable system. Similarly one can find a region around COR = 1 in which the effect of increasing the average lifetime of capital changes from being stabilizing to being destabilizing.

A point of neutral stability is a point where the eigenvalues λ_1 and λ_2 of the equations of motion (1) and (2) cross the imaginary axis from the left half-plane (negative real values) to the right half-plane (positive real values). In mathematical literature this phenomenon is known as a Hopf-birfurcation[10]. At the cross-over point the period of oscillation as determined from the imaginary parts of the eigenvalues is given by

$$T = \frac{2\pi}{|Im\{\lambda\}|} = \frac{2\pi ALC}{\sqrt{\frac{\beta}{DDC}(ALC-COR)}}$$

T is not directly proportional to the average lifetime of capital, as one might perhaps have expected. The nonlinear dependence of T on ALC is also demonstrated in the full simple model of the long wave [1]. It is worth noticing that the period of oscillation at the cross-over point is independent of the parameter α. For typical values of the other parameters we find $T \cong 30$ years. There is no (general) mathematical theory that allows us to determined the period for the fully developed limit cycle.[11]

GLOBAL STABILITY ANALYSIS

The purpose of a global stability analysis is to investigate the model behaviour well away from equilibrium. Then we have to consider the non-linear functions in more detail. Typical questions to address are:

(i) under which conditions will the non-linearities be strong enough to contain the system and produce a limit cycle?

(ii) can there be more than one limit cycle?

(iii) which parameters determine the form of the limit cycle in various regions of phase-space?, and

(iv) how fast is the limit cycle approached by a trajectory starting at a point away from the attractor?

We shall not concern ourselves with all of these problems in this paper. We would like to discuss, however, how the form of the limit cycle is controlled by certain characteristic parameters of the two non-linear functions CUF and MDP. In particular we shall investigate the significance of the assumption of a maximum to the multiplier on capital order rate from desired production at high values of DP/PO (see figure 1).

We first notice that the ratio of desired production to potential output DP/PO in the base case varies along the limit

cycle from less than 0.3 during an economic downturn to more than 2.8 in the recovery phase. The turning points of the limit cycle are therefore controlled by the behaviour of the table functions in the two regions DP/PO $\rightarrow$ 0 and DP/PO > 2. As shown in figure 6, we have introduced the following 4 parameters to describe this behaviour (the slope of the capital utilization factor CUF for DP/PO $\rightarrow$ 0 is assumed to be 1):

- a the highest attainable capital utilization factor. By definition $a > 1$, base case value $a = 1.20$.
- b the value of the multiplier from desired production on capital order rate for DP/PO = 2. By definition $b > 1$, base case value $b = 4.0$.
- G the slope of the function MDP for DP/PO > 2. Base case value $G = 0$.
- γ the slope of the function MDP for DP/PO 0. Base case value $\gamma = 0.5$.

Figure 6

Note that since the independent variable z = DP/PO in the two non-linear functions is expressed by eq. (3a), the relation z = constant is represented by a set of straight lines in the phase-plane. Figure 7 shows the lines corresponding to $z = 0.3$, 0.6, 1 and 2, respectively. With piecewise linear table functions as in our DYNAMO-model, and with a common independent variable for CUF and MDP, the phase-plane thus divides into segments within which the system has a linear representation. This facilitates the global analysis enormously.

Figure 7

To continue our analysis we determine some of the isoclines of the problem. An isocline is a curve in phase-space along which all trajectories have a given slope (direction). The 0-isocline, for instance, connects points in which dUOC/dt = 0, so that all solutions to our equations of motion (regardless of the starting point) proceed horizontally in phase-space. By solving the condition dUOC/dt = 0 for $z < .6$, we find the 0-isocline in this region to be the straight horizontal line

$$\frac{UOC}{DPG} = \frac{\gamma \cdot DDC \cdot COR}{ALC - \gamma \cdot COR} .$$

Above this line dUOC/dt is negative, and below the line dUOC/dt is positive. On both sides, the trajectories are thus attracted by the 0-isocline which like a narrow gorge guides them towards the bottom of economic depression. The position of the 0-isocline controls the smallest value of unfilled orders for capital attained during a cycle. Of the four table function parameters a, b, G and γ, only γ affects this minimum. The higher the value of γ ($0 \leq \gamma < 1$), the higher the minimum backlog of capital orders will be, and the easier it will be to initiate a new upswing.

The ∞-isocline connects points in phase space in which dPC/dt = 0, so that the trajectories are vertical. For DP/PO > 2,

111 the ∞-isocline is found to be the vertical line

$$\frac{PC}{DPG} = \frac{COR \cdot ALC}{a \cdot ALC - COR} .$$

On both sides of this line, the slopes of the trajectories point away from it. As a mountain ridge, the ∞-isocline thus separates the phase plane into two regions. A trajectory starting to the left of the ∞-isocline will never reach the limit cycle. The production capital will be too low to satisfy the orders for goods, reinvestments are impossible, and the capital sector will inevitably collapse.

The ∞-isocline sets a lower limit to the production capital during a Kondratieff-cycle. Of the four table function parameters, this minimum is determined only by a, the maximum capital utilization factor. As one would intuitively expect, the greater the ability to increase capital utilization beyond one, the lower the minimal production capital can be.

The phase-portrait of figure 7 also shows the ∞-isocline for $z < 1$ as well as the -1-isocline. The latter connects points in which (dUOC/dt)/(dPC/dt) = -1 and coincides with the z=1 line. Finally for $z > 2$ and G= 0, we have calculated the slope of a trajectory at an ordinary point in phase-plane to be

$$\frac{dUOC}{dPC} = \frac{(b \cdot COR - a \cdot ALC)(PC/DPG) + COR \cdot ALC}{(a \cdot ALC - COR)(PC/DPG) - COR \cdot ALC} .$$

Establishing the isoclines allows us to consider whether variations in the non-linear functions can create new bifurcation points in which for instance exponential growth or decline re-place the limit cycle behaviour. One might expect, for instance, that small values of a and γ could cause the trajectories to shoot underneath the left ∞-isocline into the region of economic collapse. Under such conditions, the economy would not be able to initiate a new upswing, once a downturn has started. By simulating with the model we have found this to be the case for instance for a = 1.10 and γ = 0.1.

A second possibility is that the capital build up during the initial phases of an upswing continues to accelerate and never falls back across the line z = 2 in phase-plane. Under these conditions, desired production continues to be larger than twice the potential output, and sustained exponential growth results. Both analytically, and by experimenting with the model we have found that this can occur if the slope G of the table function MDP exceeds the value

$$G \geq \frac{DDC[2a \cdot ALC-(b+1)COR]}{COR(COR-DDC)} .$$

It is clear from the above result that the threshold for the bifurcation from limit cycle behaviour to sustained growth can be relatively low if delivery delay for capital is low, if capital/output-ratio is high, and if at the same time the parameter b in the table function for the multiplier MDP is high. Even G = 0 will not always be sufficient to contain the development and produce a limit cycle, namely if COR/ALC > 2a/(b+1). The possibility of explosive behaviours sheds some light on the significance of the saturation in the multiplier on capital order rate from desired production.

THE LOCAL FLOW OF TRAJECTORIES

By means of a local stability analysis we shall now examine the convergence or divergence of nearby trajectories in phase-plane. For a stable system, the solutions of the equations of motion obtained with two neighbouring starting points converge over time. For a chaotic system, two trajectories started infinitesimally apart generally diverge. Finally, in the presence of a limit cycle, the trajectories converge towards the limit cycle, but along the attractor they tend to maintain a constant separation in time.

To describe the local stability properties of our non-linear system we again use the Jacobian (or functional) matrix, this time for a general point in the phase-plane

$$\underline{\underline{J}}(x,y) = \begin{pmatrix} P'_x(x,y) & P'_y(x,y) \\ Q'_x(x,y) & Q'_y(x,y) \end{pmatrix}$$

with

$$P'_x(x,y) = \frac{f(z)}{c} + \frac{x}{c}\frac{\partial f}{\partial z}\frac{\partial z}{\partial x} - \frac{1}{\tau} , \qquad (5)$$

$$P'_y(x,y) = \frac{x}{c}\frac{\partial f}{\partial z}\frac{\partial z}{\partial y} , \qquad (6)$$

$$Q'_x(x,y) = \frac{g(z)}{\tau} + \frac{x}{\tau}\frac{\partial g}{\partial z}\frac{\partial z}{\partial x} - \frac{f(z)}{c} - \frac{x}{c}\frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \qquad (7)$$

and

$$Q'_y(x,y) = \frac{x}{\tau}\frac{\partial g}{\partial z}\frac{\partial z}{\partial y} - \frac{x}{c}\frac{\partial f}{\partial z}\frac{\partial z}{\partial y} , \qquad (8)$$

As before f(z) and g(z) represent the non-linear functions for the capital utilization factor CUF and the multiplier from desired production MDP, respectively. $\partial f/\partial z$ and $\partial g/\partial z$ denote the derivatives of these functions with respect to their common independent variable $z = DP/PO$. $\partial z/\partial x$ and $\partial z/\partial y$ denote the partial derivatives of z with respect to the two normalized state variables x = PC/DPG and y = UOC/DPG.

The solutions $e^{\lambda_+ t}$ and $e^{\lambda_- t}$ of the locally linearized equations of motion can now be obtained from the eigenvalues $\lambda_+(x,y)$ and $\lambda_-(x,y)$ of the Jacobian matrix. These eigenvalues are known as Lyapunov exponents[10,11]. Since for an autonomous system trajectories do not cross (except at singular points such as the equilibrium point), the two eigenvalues must both be real. λ_+ then denotes the larger of the eigenvalues. As indicated by the notation $\lambda_+(x,y)$ and $\lambda_-(x,y)$, the Lyapunov exponents will vary along a trajectory.

The solution corresponding to the larger of the two Lyapunov exponents will always dominate in the long run. If $\lambda_+(x,y)$ is negative, the trajectories locally converge. Along a limit cycle, the largest Lyapunov exponent is zero on average, while at the same time the trace of the Jacobian matrix $Tr\underline{J} = P'_x + Q'_y = \lambda_+ + \lambda_-$ is negative. $Tr\underline{J}$ is equal to the divergence of the vector field $\{P,Q\}$, and requiring the trace of the Jacobian matrix to be negative is equivalent to saying that there shall be a general contraction in phase-space.

With P'_x, P'_y, Q'_x and Q'_y as given by Eq. (5-8), the eigenvalues of the two dimensional Jacobian maxtrix can again be found from (4). If we introduce two new table functions $DCUF = \partial f(z)/\partial z$ and $DMDP = \partial g(z)/\partial z$ to represent the slopes of the table functions CUF = f(z) and MDP = g(z) it is a simple matter to construct a DYNAMO-program which will calculate the Lyapunov exponents along the trajectory during a simulation.

The result is shown in figure 8. Here we have plotted the development of the normalized state variable x = PC/DPG together with two curves representing rerpectively $\lambda_+ t$ and $(\lambda_+ + \lambda_-)t$. From the slope of the first of these two curves we can read the average value of the larger of the two Lyapunov exponents. From the slope of the second curve we can read the average sum of the two eigenvalues $\lambda_+ + \lambda_-$ which also gives the divergence of the vector field $\{P,Q\}$. It is seen that in average $\lambda_+ = 0$ and $\lambda_+ + \lambda_- < 0$ in agreement with the afore mentioned characteristic of a limit cycle.

Figure 8

CHAOTIC BEHAVIOUR

Chaos[12] denotes a distinct mode of behaviour in the same way 'exponential growth' and 'damped oscillations' are characteristic behaviour modes of dynamical systems. Chaos differs from these other modes of behaviour by the fact that it can occur only in non-linear systems. Chaotic behaviour can therefore not be understood by generalization of concepts from linear systems theory.

Chaos may be characterized as a behaviour which is bounded in state space and seems to have a certain recurrence. Each swing is unique, however, the system never repeats itself, and the true period is infinite. Chaos is also characterized by the fact that while the trace of the Jacobian matrix still is negative (general contraction in state space), the highest real value for the Lyapunov exponents is positive. There is therefore an exponential divergence in one direction. This gives rise to a sensitivity both to the initial conditions and to the accuracy of the numerical calculation, not encountered for stable systems. For a chaotic system one can therefore not make point predictions, neither with a DYNAMO-model nor with any other numerical technique. Nevertheless, DYNAMO can often be used to get an impression of the general behaviour of a chaotic system.

Chaos can not occur in continuous, autonomous systems unless there are three or more state variables. However, chaotic behaviour can result, if a two-dimensional system as our simple Kondratieff-wave model is driven exogenously for instance, with a sine-wave[15].

To illustrate this possibility, we introduce a small sinusoidal variation in the desired production of goods, represented in DYNAMO-formulation by

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DPG.K = 1+AMP*COS(6.283*TIME.K/PER).
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Here $6.283 \cong 2\pi$, PER = 5 years is the period of the exogenous excitation, and the parameter AMP is the amplitude of the excitation. The above modulation of the desired production of goods can be thought of as representing the business-cycle, and we are thus considering the effects that the business cycle may have upon the Kondratieff-wave.

As shown in the phase-plots of figures 9 and 10, as the amplitude AMP of the sinusoidal disturbance is increased from 0 to 20% of the mean value of DPG, the limit cycle of the undistorted Kondratieff-wave model is transformed through a series of period doublings into chaos. Changing AMP thus causes the system to go through a series of bifurcations each of which qualitative-

ly alters the characteristic mode of behaviour. Beyond the threshold for chaos, we have found windows in which odd multipla of the fundamental period can be observed.

Figure 11 shows the development in time of the normalized state variables x = PC/DPG and y = UOC/DPG with chaotic behaviour. Note the irregular variation for instance in the peaks of the production capital. Finally figure 12 shows the same variables plotted together with $\lambda_+ t$ and $(\lambda_+ + \lambda_-)t$. On average λ_+ is now clearly positive, indicating the model's sensitivity to the initial conditions.

Figure 9

Figure 10

Figure 11

Figure 12

CONCLUSION

We have shown how a relatively simple System Dynamics model developed to simulate real world behaviour can exhibit a varity of complex dynamic modes depending upon the values of various parameters. Four different types of bifurcations have been identified and we have shown that the model can give chaotic behaviour if driven exogenously. Even for a model with two level variables and two table functions, the number of significant parameters can be so high that one can hardly hope to understand the dynamics of the model from simulation experiments alone. Under these conditions, application of various methods of formal stability analysis can be very helpful.

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The linear and global stability analyses of the one-sector Kondratieff model were intially performed during a leave of absence at Dartmouth College in the summer of 1982. Steen Rasmussen and Erik Mosekilde would like to express their gratitude to the staff of the Resource Policy Center for a very pleasant and inspiring stay.

Our preliminary investigations of chaotic behaviour were performed on the analog-digital computer MOSES, developed by Kaj Jensen at the Technical University of Denmark. MOSES is a modular simulator, where each element in a System Dynamics flow-diagram is represented by a 2x4 inch box containing a microprocessor and suitable controls. By connecting different boxes through a permanent switch-board, any SD-model can be set-up. During this process and during subsequent parameter variations, the model is automatically simulated 50 times per second, and the results are continuously shown on a colour TV-screen. The advantage of this system is that one can investigate alternative model formulations and scan through a parameter space much faster than with any digital computer. We would like to thank Kaj Jensen for the opportunity to use the MOSES system.

Finally we would like to thank Jan Frøyland, University of Oslo for pointing out to us an easy way to calculate Lyapunov exponents for low dimensional systems.

Appendix A

Our analysis of the simplified Kondratieff-wave model heavily rests upon details in the model formulation. For this reason we shall reproduce the complete DYNAMO-program for the model here:

```
*  ONE SECTOR KONDRATIEFF MODEL
L  PC.K=PC.J+(DT)(CAR.JK-DR.JK)
N  PC=PCI
N  PCI=(DPG*COR*ALC)/(ALC-COR)
R  DR.KL=PC.K/ALC
C  ALC=20 years
R  CAR.KL=PR.K-DPG
C  DPG=1.0 unit/year
L  UOC.K=UOC.J+(DT)(OR.JK-CAR.JK)
N  UOC=(PCI*DDC)/ALC
R  OR.KL=(PC.K/ALC)*MDP.K
A  PO.K=PC.K/COR
C  COR=6 years
A  PK.K=PO.K*CUF.K
A  CUF.K=TABHL(CUFT,DP.K/PO.K,0,2,.2)
T  CUFT=0/.2/.4/.6/.8/1/1.1/1.15/1.18/1.19/1.20
A  DP.K=(UOC.K/DDC)+DPG
C  DDC=3 years
A  MDP.K=TABHL(MDPT,DP.K/PO.K,0,2,.2)
T  MDPT =0/.1/.2/.3/.5/1/2/3/3.5/3.9/4.0
SPEC  DT=.2
```

With the above specification the model is started in equilibrium. In the runs we have performed, either the model has been started out of equilibrium by specifying particular initial values to the two state variables, the model has been excited by a step input to desired production, or the model has been driven by a small amplitude sinusoidal variation in the desired production of goods.

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Figure captions

Figure 1. System Dynamics flow-diagram for a simplified one-sector Kondratieff-wave model.

Figure 2. Simulation results for the simplified Kondratieff-wave model with a capital/output-ratio of COR=6. The model exhibits a limit cycle behaviour. (a) time-development, and (b) phase-plot.

Figure 3. Similar results as in figure 2 only with a capital/output-ratio of COR=4. The model now exhibits damped oscillations towards a stable equilibrium.

Figure 4. The parameters α and β are defined as the slopes of the two table functions CUF and MDP, respectively, in the equilibrium point (1,1).

Figure 5. Neutral stability curves for the simplified Kondratieff-wave model: (a) with α as a parameter and fixed values for ALC and DDC, and (b) with DDC as a parameter and fixed values for α and ALC.

Figure 6. This figure defines the four parameters a, b, G and γ used in the global stability analysis of the Kondratieff-wave model.

Figure 7. Phase-portrait of the Kondratieff-wave model. The portrait shows the unstable equilibrium point, the limit cycle attractor, and a set of different isoclines with their corresponding trajectory tangents. The figure also shows how the phase-space is divided into segments within which the system is linear.

Figure 8. Simulation results obtained with normal parameter values. Besides the development of the two normalized state variables PC/DPG and UOC/DPG, we have also plotted the functions $\lambda_+ t$ and $(\lambda_+ + \lambda_-)t$. λ_+ and λ_- are the Lyapunov exponents of the problem. In average over the limit cycle $\lambda_+ = 0$ and $\lambda_+ + \lambda_- < 0$.

Figure 9. With a 5.0% sinusoidal modulation of the desired production of goods, the Kondratieff-wave model shows period doubling.

Figure 10. With 20% modulation of the desired production of goods, the Kondratieff-wave model shows chaotic behaviour. (Phase-plot corresponding to the time-plot of figure 11).

Figure 11. With 20% modulation of the desired production of goods, the Kondratieff-wave model shows chaotic behaviour.

Figure 12. Same results as in figure 8 only with a 10% modulation in the desired production of goods. The system is now chaotic, and the larger Lyapunov exponent is positive in average.

Figure 1

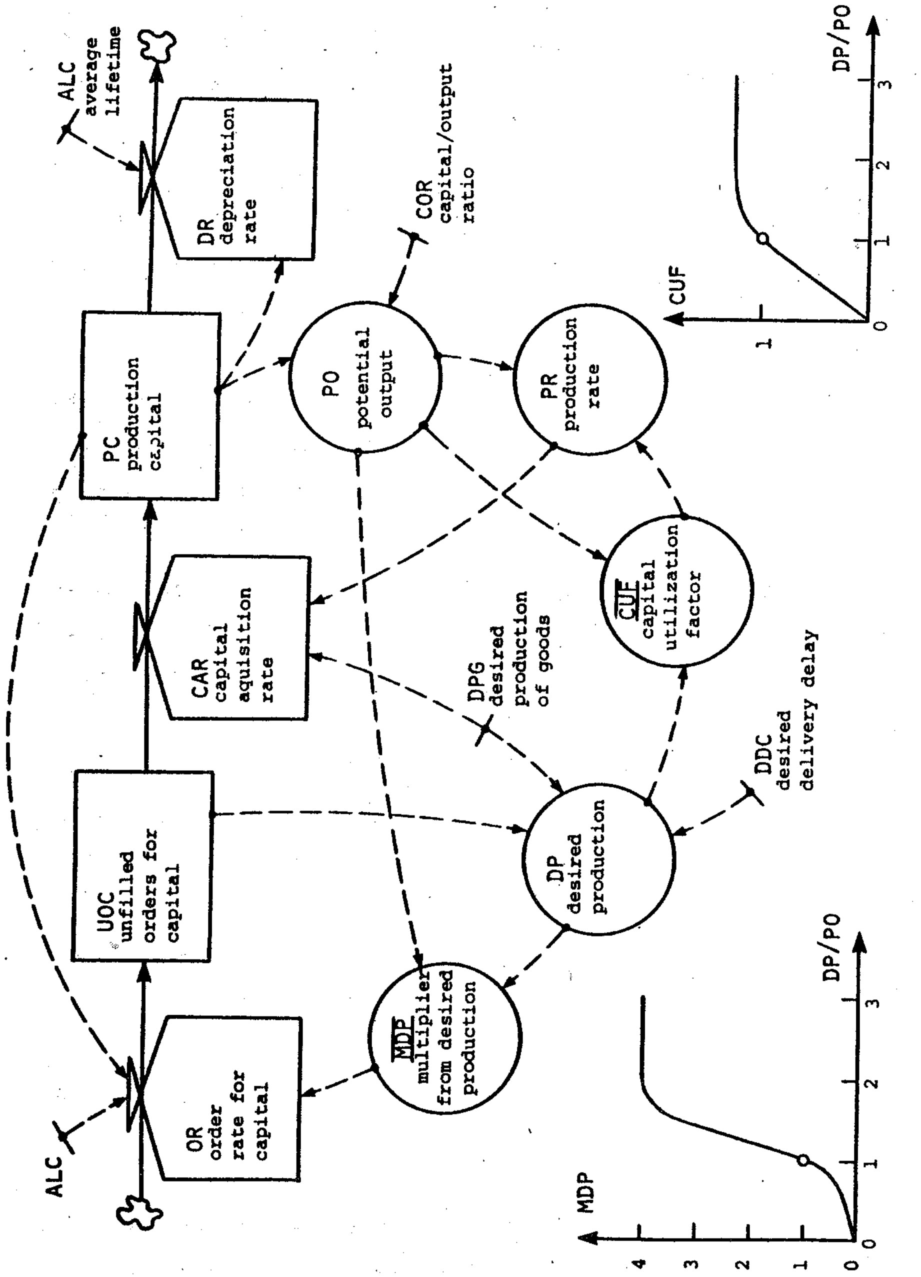

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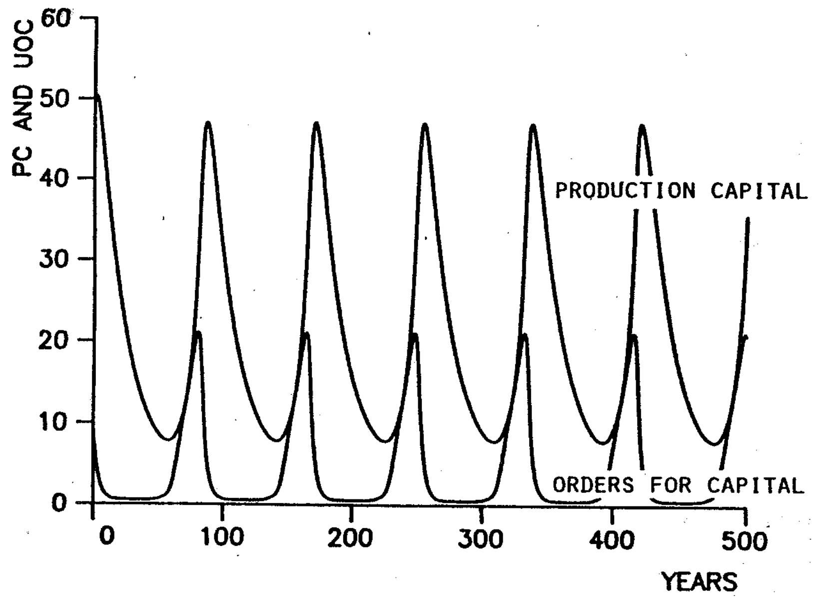
Figure 2a



ALC=20, DDC=3, COR=6

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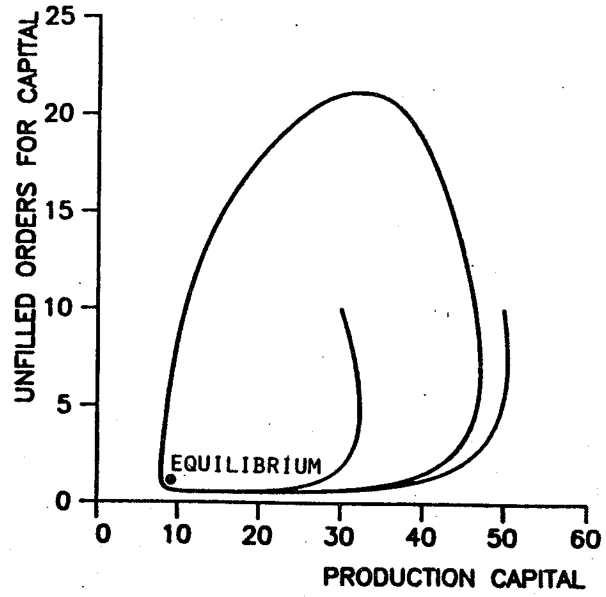
Figure 2b



ALC=20, DDC=3, COR=6

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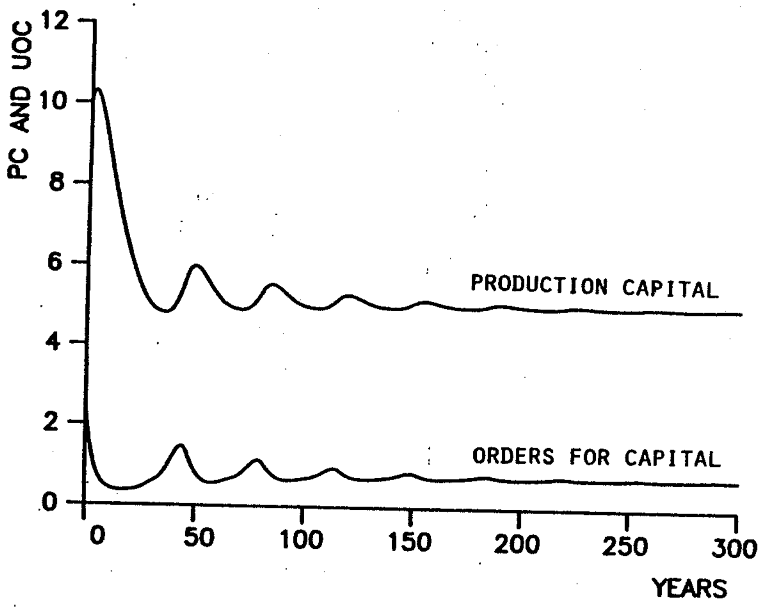
Figure 3a



ALC=20, DDC=3, COR=4

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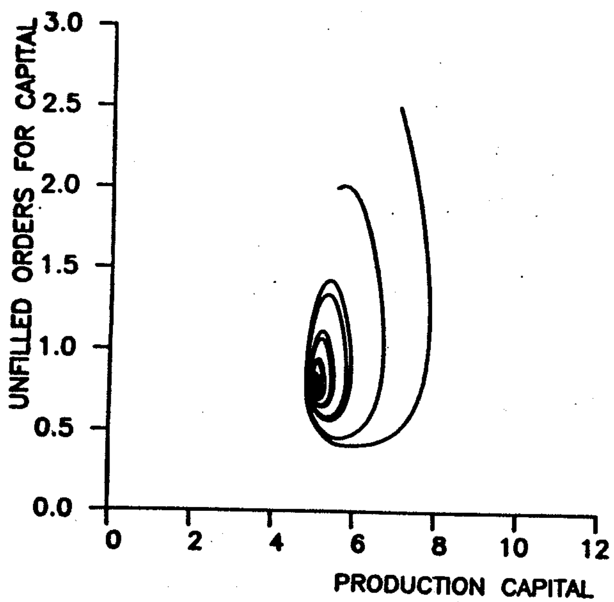
Figure 3b



ALC=20, DDC=3, COR=4

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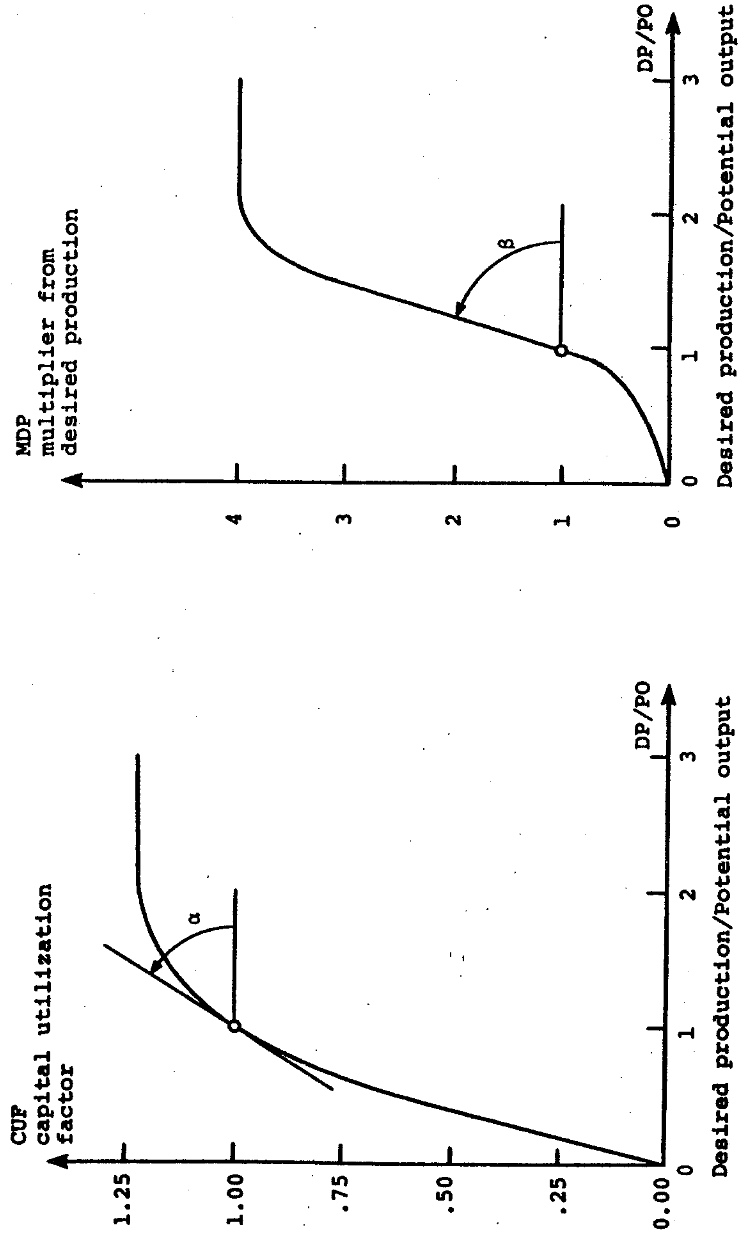
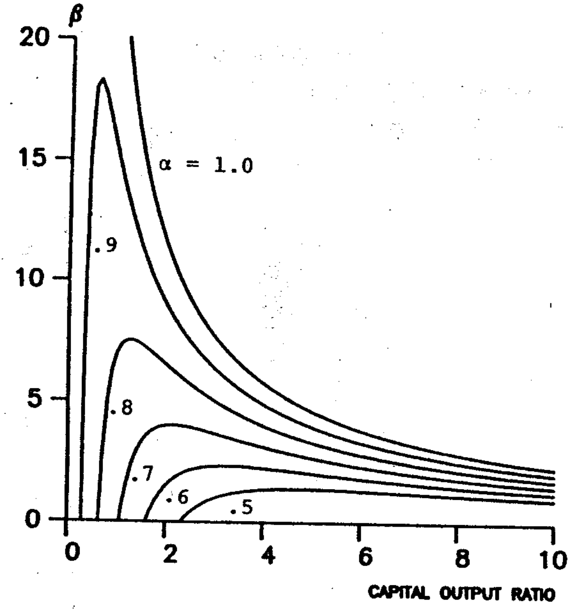
Figure 4





ALC = 20 YEARS, DDC = 3 YEARS

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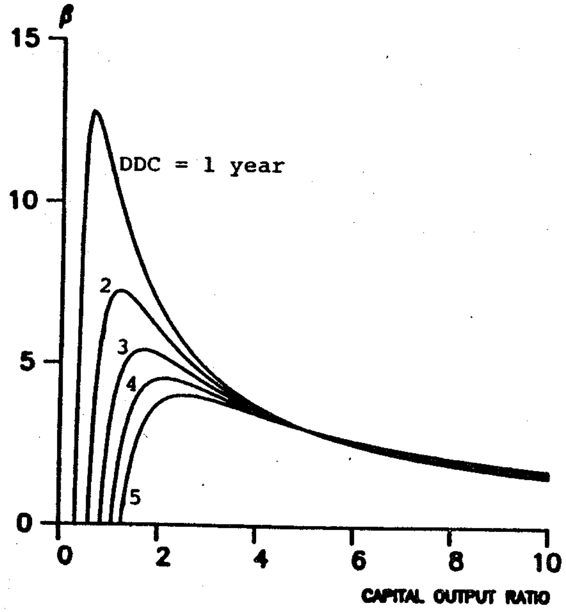
Figure 5a



α = 0.75, ALC = 20 YEARS

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Figure 5b

Figure 6

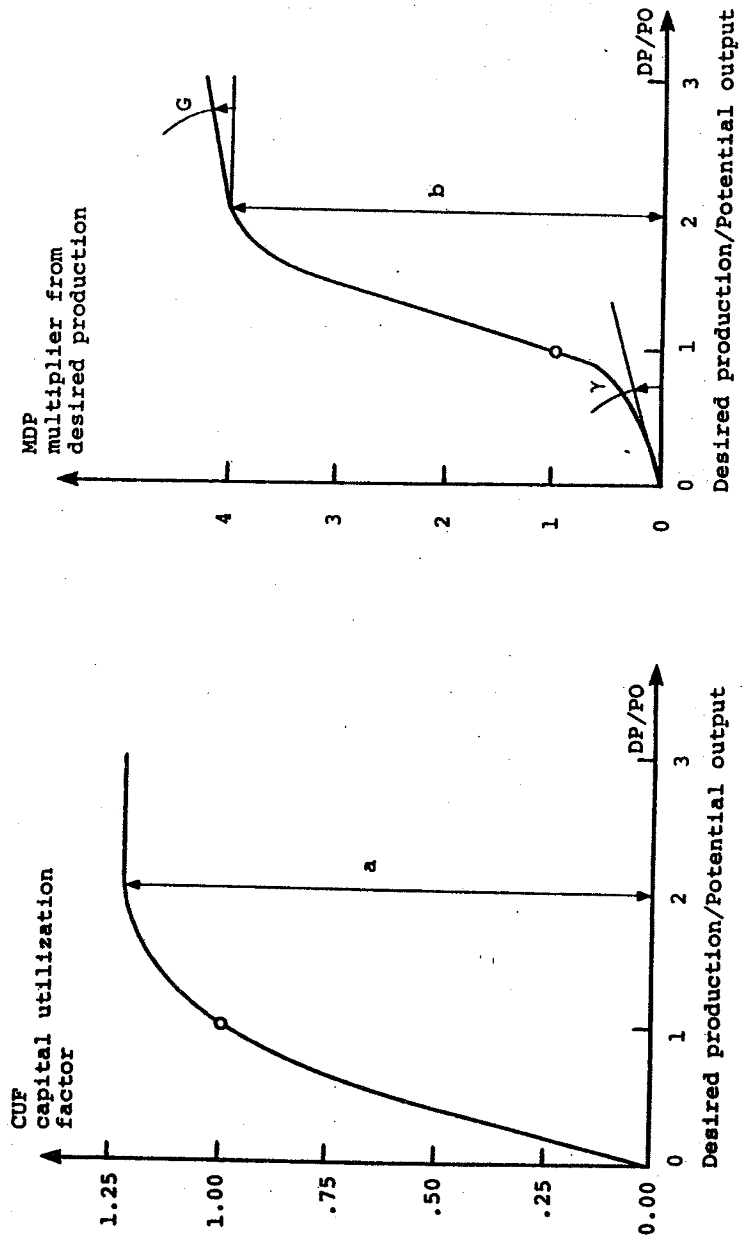

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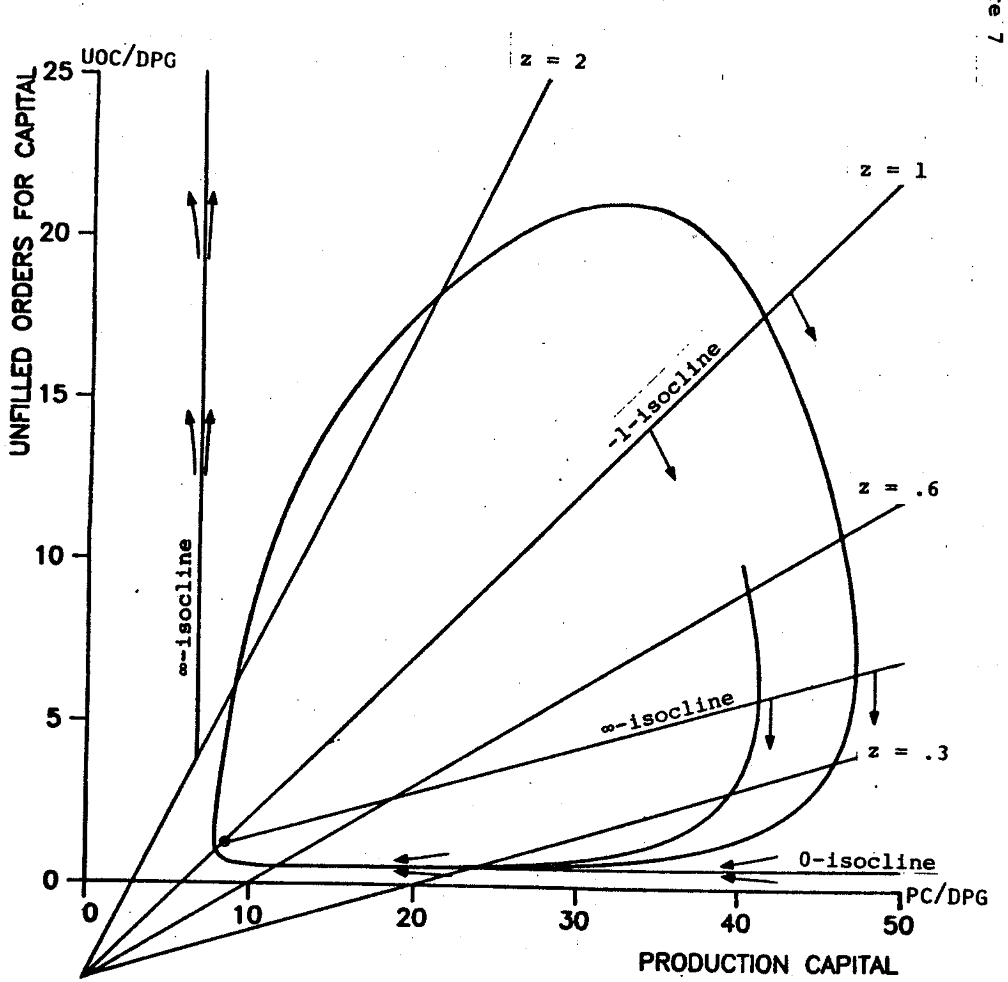
Figure 7



ALC=20, DDC=3, COR=6

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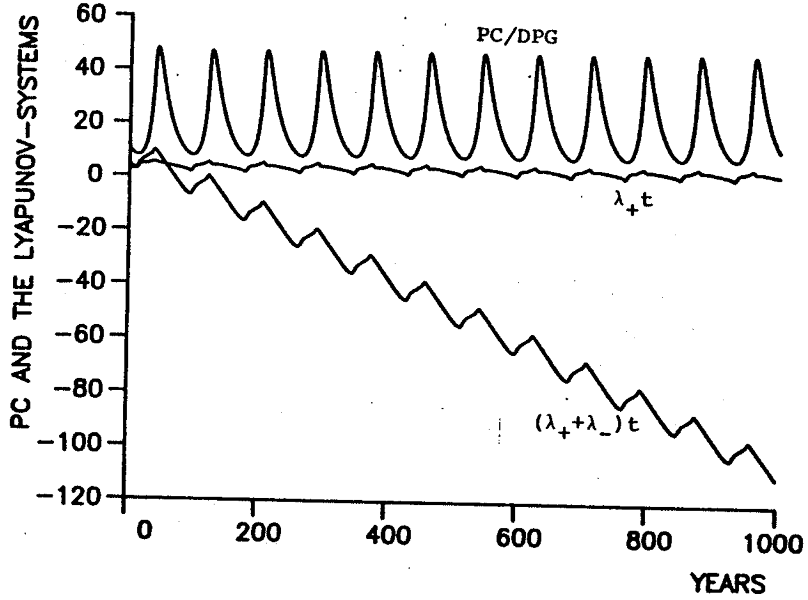
Figure 8



ALC=20, DDC=3, COR=6, AMP=0

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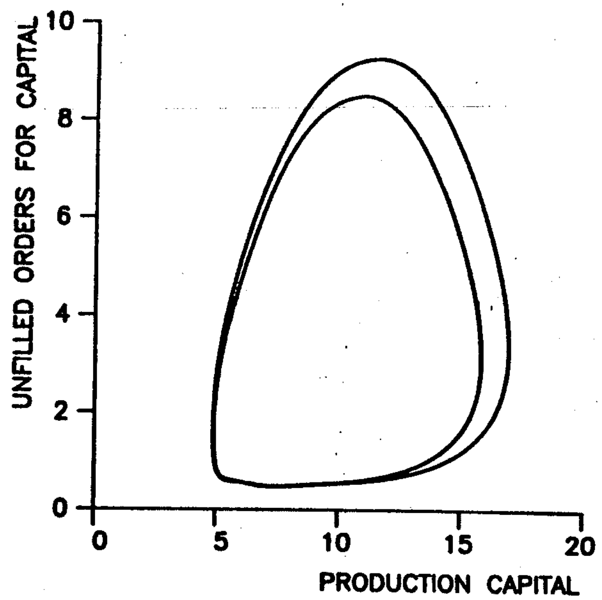
Figure 9



ALC=15, DDC=3, COR=4, AMP=0.05, TIME=10

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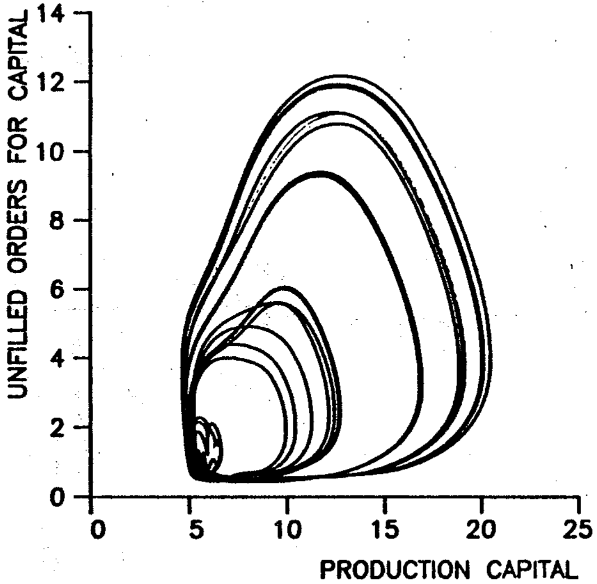
Figure 10



ALC=15, DDC=3, COR=4, AMP=0.2, TIME=10

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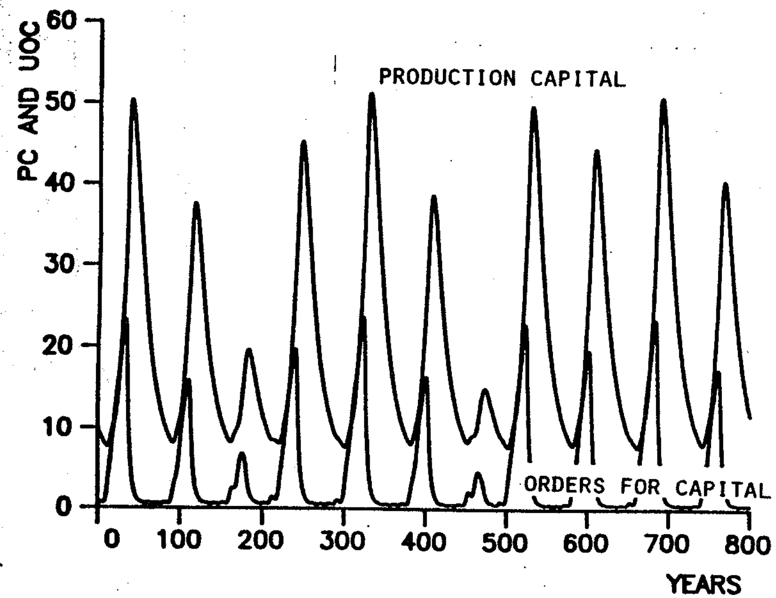
Figure 11



ALC=20, DDC=3, COR=6, AMP=0.2, TIME=10

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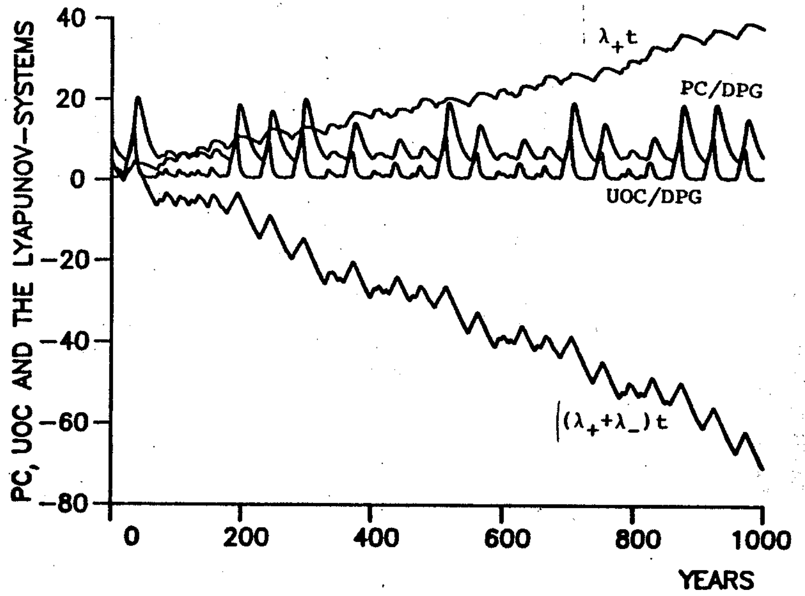
Figure 12



ALC=15, DDC=3, COR=4, AMP=0.2, TIME=10