THEORETICAL APPROACH TO LONG TERM COMPANY BEHAVIOUR

H.Sedehi, R.Serra, S.Vassallo
TEMA S.p.A.
Viale Aldo Moro, 38
Fiera District
40127 Bologna

ABSTRACT

In order to study the long term behaviour of complex systems, such as industrial enterprises, it is necessary to use reduced models with a limited number of variables. Here we investigate theoretically the relationship between these "mesoscopic" models and more detailed, "microscopic" models of the same physical systems. When the relevant variables evolve more slowly than the irrelevant degrees of freedom, a powerful projection technique is presented (Adiabatic Elimination Procedure). A pedagogical example is discussed, dealing with a large company in the field of computer science, which wants to increase its presence in a particular market segment by starting a cooperation with a small but aggressive company,already in that market segment.

1. INTRODUCTION

Powerful digital computers have allowed the construction of models of previously unthinkably complex levels. We will not dwell on this rather trivial observation, nor on the obvious advantages of this approach. We prefer to stress the problems of interpreting a vast amount of results of computer experiments, which sometimes show very different behaviours. Moreover, as is well known, when the number of variables increases it becomes corresponsingly difficult to provide meaningful sets of initial conditions.

In order to understand a system's behaviour, it is also important to develop so-called reduced models, which give a rough description of that behaviour in terms of a small number of relevant variables. The interplay between reduced models and detailed, "microscopic" models has proved to be very fruitful in the physical sciences, especially in condensed matter physics [1] - [5] .

We think that such an integrated approach would also bear extremely useful instruments in socio-economic modelling. In this paper we will discuss the relationship between microscopic and reduced, or mesoscopic, models. The mesoscopic approach will prove to be particularly powerful when we deal with the long term behaviour of complex systems. In this case, the "relevant" variables (i.e., the more important ones) are also usually the slower ones. We are not interested in the detailed behaviour of a large se of "fast" variables, but

rather in the general trends of a few so called "order parameters", whose evolution takes place on a timescale much slower than the timescale of the evolution of the other variables [1] [2] [4]. This allows a tremendous simplification of the model equations. The main purpose of this paper is to illustrate such simplifications.

The need for long term models cannot be overemphasized, especially in turbulent times where decisional aids are necessary. The techniques we discuss here were born in the physical sciences, where similar problems are encountered when dealing with macroscopic systems far from thermodynamic equilibrium. While avoiding a naive reductionist approach, we think it important to exploit the contribution of such techniques to socio-economic modelling.

We will also discuss an example, mainly of a demostrative nature. The example deals with a big company (E), operating in computer science inside a large corporation, whose goal is to improve its presence in the external market (namely, companies not belonging to the same corporation). In order to achieve this goal, a large reorganization must take place. However, it is the management's convinction that such a goal cannot be accomplished without cooperation with a smaller but more aggressive company (T). Such cooperation may be an acquisition of T, a joint venture, or some other arrangement. The main purpose is to achieve the commercial and product management capabilities of T. Such an effort would involve:

- commercial reorganization (creation of a central marketing organization and a large commercial network); above all, creation of customer-oriented thinking;
- production reorganization (especially with regard to project management, budgeting, a.s.o.);
- R & D reorganization, with particular emphasis on innovation and new product development.

The acquisition of T may or may not render the transition to a wider presence on the market segment of interest possible. The oversimplified model we will discuss focuses its attention on this transition, which is assumed to be analogous to the transitions which are encountered in bistable systems.

Finally, in the conclusions, emphasis is again placed on the need to use the microscopic and mesoscopic approaches in a complementary way.

2. ADIABATIC ELIMINATION OF RAPIDLY RELAXING VARIABLES

Let us consider a dynamical system:

(1) $\dot{x} = Gx$

Where x is an N-dimensional vector and G is, in general, a non linear differential operator. This is a deterministic system, whose time evolution is completely determined by a given set of initial conditions:

(2) $x(0) = x_o$

However, our knowledge of a given physical or economic system does not allow us, in principle, to use strictly deterministic equations. There are two main reasons for this: first, our system may interact with an environment which we have decided not to describe in detail, but which affects its evolution. Second, an "exact" or "microscopic" model of our system might need a huge number of variables, so we always use some sort of reduced description [1] [4] [6]. We resort therefore to a statistical description of our system. Let us define a probability density $p(x,t) = p(x_1,x_2 \ldots x_N,t)$, such that $p(x,t)\, dx_1 \ldots dx_N$ is the probability that the state variables lie, at time t, between x_j and x_j+dx_j, $j=1,\ldots.N$.

If the time evolution of the state variables is described by Eq.(1), then the time evolution of the probability density is described by:

(3) $\frac{\partial}{\partial t} p(x,t) = Lp(x,t)$

Where the operator L is the adjoint of the operator G, provided that we suitably define a scalar product in the function space of interest. Indeed, it can be proven that [6]:

(4) $\int x \left[Lp(x,0)\right] dx = \int \left[Gx\right] p(x,0)dx$

The relationship between Eq.(1) and Eq.(3) is analogous to the relationship between the so called Heisenberg and Schrödinger pictures of quantum mechanics.

We will now suppose that the set of state variables of our system (x) is divided in two subsets:

- the relevant variables, which we will call "a" variables, whose time evolution is slower than the others;

- the fast, or irrelevant variables, which will be called "b" variables, whose time evolution is much faster then the time scale of interest.

In the following we will use the same formalism to deal with both the case of an open system, where the variables fluctuate because of the coupling with their environment, and the case of a closed system.

Before describing a systematic procedure for achieving the desired reduced description (Adiabatic Elimination Procedure), we will discuss the fundamental concepts in a simple example [1]. Let us consider the one-dimensional equation:

(5) $\dot{x} = -cx + F(t)$

which can describe, for instance, an overdamped oscillator or, in general, a system which would tend towards the equilibrium value x=0 if not disturbed, but which is perturbed by the external force F(t). The solution of Eq.(5), with x(0) = 0, is:

$$(6) \quad x(t) = \int_0^t e^{-c(t-s)} F(s)ds$$

The value of the variable x at time t then depends upon the past history of F. Now let us assume that

$$F(t) = ke^{-dt}$$

From Eq.(6) we then get (supposing also that the intrinsic decay rate of x is much faster than F, i.e., $c \gg d$):

$$(7) \quad x(t) = \frac{k}{c-d}\left[e^{-dt} - e^{-ct}\right] \cong \frac{F(t)}{c}$$

We might have obtained the same result by formally putting x=0 in Eq.(5). That is indeed the simplest way to eliminate the fast variables, although sometimes it is too simple. The time derivate of the fast variables is put equal to zero, because their dynamic properties refer to a time scale much faster than the time scale of interest. Since we are interested only in the time evolution of the order parameters, we suppose that the fast variables instantaneously adjust themselves to the equilibrium values which correspond to a given set of values for the order parameters. This simple prescription is useful when the separation between the time scales of the relevant and irrelevant variables is very large.

Let us now turn to a systematic method of reducing the number of variables (Adiabatic Elimination Procedure, AEP) [4] [6] . It is always possible to divide the dynamic operator L as follows:

$$(8) \quad L = L_o + L_{ab}$$

$$(9) \quad L_o = L_a + L_b$$

where L_a (L_b) acts only upon the a (b) variables, while L_{ab} is composed of the mixed interaction terms. We define the probability density for the relevant variables as follows:

$$(10) \quad s(a,t) = \int p(a,b,t)db$$

We also define the projection operator P for the set of relevant variables as:

$$(11) \quad P = p_{eq}(b) \int db$$

where $p_{eq}(b)$ is the asymptotic distribution of the b variables in the case of no interaction with the relevant variables. It is mathematically defined by:

$$(12) \quad L_b p_{eq}(b) = 0$$

It can then be shown that Eq.(3) can take the following form [4][6] :

$$(13)\ \frac{\partial}{\partial t} P\tilde{p}(x,t) = PL_{ab}(t)P\,\tilde{p}(x,t) + PL_{ab}(t)\overleftarrow{\exp}\left[\int_0^t QL_{ab}(s)ds\right] Q\tilde{p}(x,0) + \int_0^t PL_{ab}(t)\ \overleftarrow{\exp}\left[\int_\tau^t QL_{ab}(s)ds\right] QL_{ab}(\tau)P\,\tilde{p}(x,\tau)\,d\tau$$

with

$$(14)\quad Q = 1-P$$

$$(15)\quad \overleftarrow{\exp}\left[\int_{t_0}^t A(s)ds\right] = 1 + \int_{t_0}^t A(s)ds + \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2\ A(t_1)\ A(t_2)\ldots$$

$$(16)\quad \tilde{p}(x,t) = e^{-L_0 t}\ p(x,t)$$

$$(17)\quad L_{ab}(t) = e^{-L_0 t}\ L_{ab}\ e^{L_0 t}$$

Those familiar with quantum mechanics will easily identify the latter equations as the equations of the interaction representation.

The results we have obtained so far are exact, and no approximation has yet been introduced. We now consider the three terms on the right hand side of Eq.(13). The first often vanishes identically, and this will also be the case in our example, so it will no longer be considered.
The second term comes from the application of a dynamic operator to the initial distribution p(x,0). It vanishes if $p(x,0) = p_{eq}(b)$ times an arbitrary function of only the relevant variables. In our example we will consider such an initial condition, so that we can also forget this "preparation" term. It must be remembered, however, that, unless we are dealing with a suitable initial condition, this term cannot be neglected.

We finally consider the third term on the r.h.s. of Eq.(13). It can be shown that the integrand depends only upon the time difference $t-\tau$. So, recalling def. (10), we obtain

$$(18)\quad \frac{\partial}{\partial t} s(a,t) = \int_0^t K(t-\tau)\, s(a,\tau)\, d\tau$$

$$(19)\quad K(t-\tau) = \frac{1}{P_{eq}(b)}\ PL_{ab}(t)\overleftarrow{\exp}\left\{\int QL_{ab}(s)ds\right\} QL_{ab}(\tau)P\,P_{eq}(b)$$

This form is well suited to a perturbative approach. We see from Eq.(18) that the time derivative of s(a,t) depends not only upon the present value of s, but also upon the previous values. These are weighted by a memory kernel K, and it is reasonable to assume that it is a decreasing function of its argument (the system gradually "forgets the

past"). We are therefore naturally led to expand $s(a,\tau)$ around the point $\tau=t$, and to approximate Eq.(18) with the following scheme:

$$(20)\quad \frac{\partial}{\partial t}\, s(a,t) = \int_0^\infty K(t-\tau)\left[s(a,t) + \frac{\partial}{\partial t}\, s(a,t)(t-\tau)+\cdots\right]d\tau$$

Note that we have changed the upper integration limit. This does not cause any appreciable error if K is significantly different from zero only in a small interval around $\tau=t$. Moreover, we see from the definition that K has a time-ordered exponential

$$(21)\quad \overleftarrow{\exp}\left[\int_\tau^t Q L_{ab}(s)\,ds\right]$$

Since the interaction is supposed to small or, in other words, to describe a slow process, we must also expand the exponential in power series. We thus have a double power series. Here we will only write the expression obtained for the lowest perturbation order ($s(a,\tau) = s(a,t)$, $\overleftarrow{\exp}\,[\int Q L_{ab}(s)ds] = 1$) :

$$(22)\quad \frac{\partial}{\partial t}\, P\, p(a,b,t) = P\, L_{ab} \int_0^\infty e^{L_{ab}(t-\tau)}\, d\tau\, L_{ab}\, P\, p(a,b,t)$$

This formula will be used in the following example.

3. AN EXAMPLE OF ADIABATIC ELIMINATION

We will now discuss an example of adiabatic elimination of fast variables. The example has a pedagogical character, and we do not claim it to be a realistic one.

Let us consider a large company E which operates in computer science, selling the time of its employees. We suppose that this company wishes to increase its presence in a particular market segment: we may think, for istance, that E belongs to a big industrial corporation, and that it desires to increase its presence in the external market. We try to model this situation with a dynamic equation for a variable x which measures the strength of E in the market segment of interest. Let us suppose that the total number of man/hours per year is fixed, and equal to N, and let M be the number of man/hours per year in the market segment of interest. Let us then define the dimensionless variable.

$$(23)\quad x = \frac{M-(N-M)}{M(N-M)}\, N$$

where x is a growing function of M; x tends to $+\infty$ if M=N, and to $-\infty$ if M=0 and is equal to 0 if M=N/2. We now suppose that the structure of the company allows the hypothesis that there may exist two stable stationary states. The first one, which is also the initial condition, represents a limited presence in the market of interest, while the second one corresponds to a desired goal, where that market segment is

the most important one for the company. This situation can be described by the following equation:

(24) $\dot{x} = -V'(x) + f(t)$

where V is a double well potential, whose analytical expression we assume to be:

(25) $V(x) = V_o a^{-4} (x^2 - a^2)^2$

f(t) is a stochastic force, which mimics several environmental fluctuations affecting the behaviour of x. For the sake of simplicity we will assume it to be a "white" stochastic force, whose statistical properties are (the brackets $\langle \ldots \rangle$ indicate ensemple averages)

(26) $\langle f(t) \rangle = 0$

$\langle f(t)f(s) \rangle = 2D\,\delta(t-s)$

D is the diffusion coefficient, and δ is the well known Dirac delta "function". Standard techniques in the theory of stochastic processes [1] [4] [7] allow us to write the evolution equation for the probability density p(x,t), the so-colled Fokker-Planck equation:

(27) $\frac{\partial}{\partial t} p(x,t) = \left[\frac{\partial}{\partial x} V'(x) + D \frac{\partial^2}{\partial x^2} \right] p(x,t)$

Its stationary solution is:

(28) $P_{\infty}(x) = \text{cost.} \exp(-V(x)/D)$

In Fig. 1 we have shown both the potential function V(x) and the asymptotic distribuction $p_{\infty}(x)$. The two stationary states (+a and -a) have the same probability as $t \to +\infty$. This is a consequence of the detailed form of V(x), which shows two minima of equal depth. The physical interpretation is that, sooner or later, the transition will take place. This is to be contrasted with a deterministic analysis [8] based on Eq.(29):

(29) $\dot{x} = V'(x)$

According to such equation, the final state is +a or -a depending upon the fact that $x(0) > 0$ or $x(0) < 0$. Although the presence of fluctuations assures us that the transition will take place, it might take an astronomically long time. The mean first time of passage across the potential barrier is indeed proportional to $\exp(-V_o/D)$, and it grows exponentially with the barrier height, V_o. If $V_o \gg D$, then some kind of intervention is necessary in order to achieve the desired goal within a reasonable period of time.

We suppose that such an intervention consists of cooperating with a smaller company T, via joint ventures, or by directly acquiring control of T. T is already present in the market segment of interest, and is rather aggressive, although much smuller than E. Let Y be the overall number of hours which T can sell in a year (both of its employees and consultants). Also let y_o be its average value, and let $y = Y - y_o$ be the deviation from the reference value. We suppose that, before joining E, y obeys the following equation:

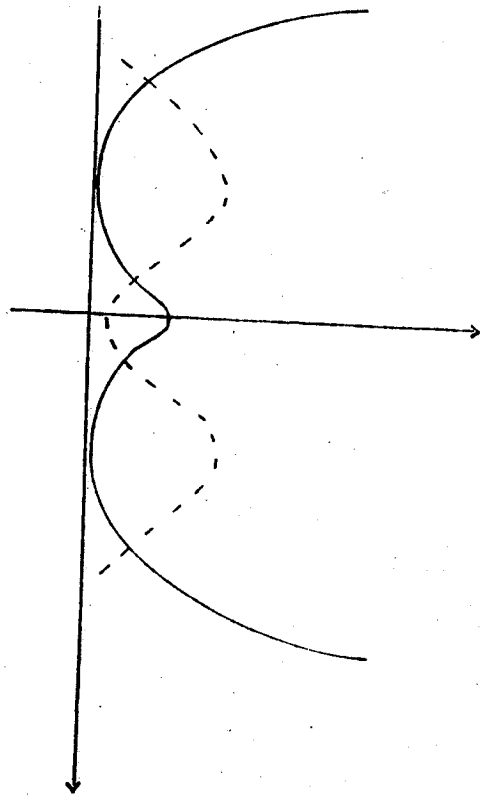

Fig. 1
Potential function V(x) and stationary probability density $P_\infty(x)$ (dashed line) after Eq.(28).

(30) $\dot{y} = -\lambda y + f'(t)$

Where f'(t) is a white stochastic force with diffusion coefficient D'.

We describe the coupling with the smaller company by adding an interaction term to the r.h.s. of Eq.(24), which we assume to be of the type -AxY (A=constant). The rationale behind this hypothesis is that the impact of the cooperation with T is proportional to the overall size of T as well as to the previous position of E in the market segment of interest, x. We do not claim that such an hypothesis is realistic: other, more accurate, expressions could be obtained by a more detailed microscopic model of our system (see conclusions). For the time being we simply want to illustrate a simple case of the AEP.

The interaction will also deeply affect the behaviour of T and there is no difficulty in introducing a term, on the r.h.s. of Eq.(30), which represents such an effect. However, once more for the sake of simplicity, we will assume that our managerial choice is to leave the dimensions of T unchanged. Under such hypotheses, the model equations become:

$$(31)\quad \begin{cases} \dot{x} = -\tilde{V}'(x) - Axy + f(t) = \tilde{d}x - bx^3 + f(t) \\ \dot{y} = -\lambda y + f'(t) \end{cases}$$

with

$$\tilde{V}(x) = -(\tilde{d}/2)x^2 + (b/4)x^4 + V_o$$

$$(32)\quad \tilde{d} = d - Ay_o$$

$$d = 4V_o/a^2$$

$$b = d/a^2$$

A first possibility is to apply the direct adiabatic elimination, putting $\dot{y}=0$. We would then obtain

$$(33)\quad \dot{x} = \tilde{d}x - bx^3 - Axf'(t)/\lambda + f(t)$$

However, there is an ambiguity in interpreting an equation like (33), where a white noise term is multiplied by a function of the state variable x (the so-called multiplicative noise) [10] . There exist in the literature two different prescriptions for associating a Fokker-Planck equation to Eq.(33), associated with the names of Itô and Stratonovich. In our case, we can apply a theorem by Wong and Zakai [11] which states that the Stratonovich algorithm should be preferred. We stress that in our case any ambiguity can be avoided by using the AEP previously described. There is indeed no difficulty in writing a Fokker-Planck equation for p(x,y,t), which takes the form of Eq.(3), and to project it onto the subspace of the x variable [4] [6] [9] . In so doing, we obtain the following equation (at the lowest

perturbation order)

$$(34)\quad \frac{\partial}{\partial t}p(x,t) = \left[-\frac{\partial}{\partial x}\left(\tilde{d}x - bx^3\right) + Q\frac{\partial}{\partial x}x\frac{\partial}{\partial x}x + D\frac{\partial^2}{\partial x^2}\right]p(x,t)$$

with

$$(35)\quad Q = D'A^2/\lambda^2$$

It does indeed coincide with the Stratonovich rule. We briefly mention that recently the group of prof. Grigolini at the University of Pisa has suggested that the Itô - Stratonovich problem should always be solved in this way, namely by resorting to a wider description of the system, and only later taking the white noise limit when applying the AEP [9] .

Let us consider the interesting limiting case $D \to 0$ (the additive stochastic forces can be neglected). The stationary solution of Eq.(34) then becomes:

$$(36)\quad P_\infty(x) = \text{const.}(x^2)^{\frac{1}{2}(\tilde{d}-Q)}\, e^{-bx^2/2Q}$$

The behaviour of $p_\infty(x)$ is shown in Fig. 2. There exists a threshold value for the parameter Q, which measures the effect of the fluctuations in T on the behaviour of E. As long as $Q < \tilde{d}$, the stationary distribution resembles that of Fig. 1. However , when $Q > \tilde{d}$ the distribution (36)peaks at the origin, where it becomes singular. This singularity would be removed if we had taken into account the small additive fluctuations. A distribution peaking at the origin represents, in our approximation, an activation effect, i.e. a jump over the potential barrier, the desired goal.

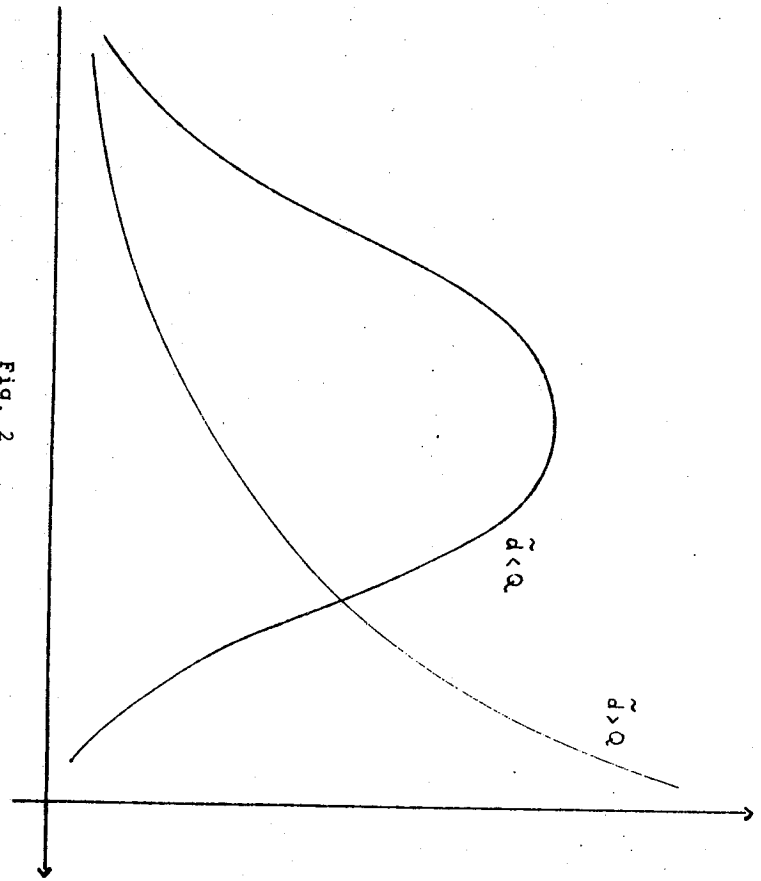

Fig. 2 Stationary probability density $p_\infty(x)$ (Eq(36)), for two different values of the parameter Q.

4. CONCLUSIONS

We have discussed the need for projection techniques, in order to reduce the number of variables in socio-economic models to a manageable level. We have also shown both a "quick" way to achieve the elimination of the faster variables and a systematic procedure which allows one to deal with cases where the separation between the time scales of the relevant and irrelevant variables is not too great. We have also discussed a demonstrative example, which shows what kind of results are to be expected using such a method.

A major problem is the relationship between a reduced description in terms of a limited number of order parameters and a more detailed "microscopic" model of the same system, of the kind familiar to system dynamicists. We believe that the two approaches are complementary. Microscopic modelling provides the detailed analysis necessary to avoid an excess of arbitrariness in preparing the reduced models. On the other hand, the use of mesoscopic models is of fundamental importance for extracting meaningful information on the system behaviour, avoiding wild "paper proliferation". The situation may be interestingly compared with the present situation in the theory of the liquid state, where molecular dynamics provides the "microscopic models", while reduced models provide the interpretation guidelines. The interplay between these two, and also with laboratory experiment, has been chosen by the European Molecular Liquids Group as

their basic methodology [5] .

We believe that adiabatic elimination techniques will prove to be very useful in socio-economic modelling in those cases where we can identify a set of relevant variables whose time evolution is slower than the time evolution of another set of variables which are not interesting in themselves, but which influence the evolution of the relevant variables. A major field of application should concern the long term behaviour of complex systems. Moreover we suggest that such techniques should prove useful in market and perhaps production modelling, rather than in modelling financial subsystems where the interest lies in a faithful and careful description of events which mainly take place in other subsystems of the firm.

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