Linear Analysis and Model Simplification

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ABSTRACT

There has been a great deal of work done in the simplification of linear dynamic models. Given that most models that are in use are nonlinear this has restricted the applicability of the available techniques. By concentrating on a particular nonlinear phenomenon, in this case shifting loop dominance, it is possible to use the techniques of linear analysis for the simplification of nonlinear models. The theory for this is developed and it is shown how this can be applied to a model. For purposes of exposition the market growth model is used and the results are encouraging. Though there is still a good deal of work to be done it seems feasible to develop simplification techniques for nonlinear models that address directly the nature of the nonlinearities.

INTRODUCTION

There has been a good deal of work done in the development of techniques for analyzing and simplifying linear dynamic models. Though the results of this work are important to the System Dynamics researcher the assumption of linearity is a serious restriction. In this paper we will look more explicitly at nonlinear models, though we will continue to use the tools of linear analysis. Since the analysis of nonlinear models is obviously a large and difficult topic no attempt will be made to deal comprehensively with the problem. Instead we will concentrate our attention on a particular type of nonlinear phenomenon, that of shifting loop dominance. In this framework we will show how the techniques of linear analysis with some extension can be useful.

We begin the paper with a brief review of the work that has been done in simplification of linear models. The essential results along with their usefulness to the modeler will be outlined. The techniques considered are strictly applicable only to linear models, though they can be applied to nonlinear models through linearization. Following this the issues of shifting loop dominance and their relationship to the results of linear analysis are discussed. We use this discussion to develop some theory for understanding and simplifying nonlinear models characterized by shifting loop dominance. These results are then applied to a relatively simple model to illustrate their usefulness. Finally, some indications of areas for future research are given.

LINEAR MODEL SIMPLIFICATION

We use the term model simplification to include a number of approaches to model analysis. In many cases the approaches simply involve trying to get some understanding of what structures in a model determine certain of the behavior modes generated by the model. It will not always be the case that an explicit simplified model will be built. Determining elements of the structure generating behavior is, however, the equivalent of building an implicit simplified model. The explicit process of simplification can be thought of as taking this sort of model analysis one step further. We are not advocating explicit model simplification in all cases, but simply pointing out that the processes of analysis and simplification are closely related.

The results on model simplification are applicable to linear dynamic models of the form

$$\dot{\underline{x}} = \underline{Ax} + \underline{Bu} + \underline{\varepsilon} \qquad (1)$$

with $\underline{x}$ a vector of length N of endogenous variables, $\underline{u}$ a vector of exogenous variables and $\underline{\varepsilon}$ a white noise error process. The process of model simplification concentrates on the internally generated dynamics, essentially the $\underline{A}$ matrix. This is consistent with stressing the endogenous point of view. Simplification consists of finding a similar model with a smaller number of endogenous variables. That is, a model of the form

$$\dot{\underline{\tilde{x}}} = \underline{\tilde{A}\tilde{x}} + \underline{\tilde{B}\tilde{u}} + \underline{\tilde{\varepsilon}} \qquad (2)$$

with $\tilde{x}$ a vector of length n<N. The difference between the original and simplified model is simply one of size. The simplified model represents an explanation for part of the original model on the basis of a subset of the variables of the original model.

A simplified model will, of necessity, be different from the original model. In order to carry out a simplification it is therefore necessary to concentrate on certain characteristics of the model. The characteristics traditionally considered, and the ones we consider, are a selected subset of the endogenously generated patterns of behavior. This is a useful focus for simplification and, given the emphasis in system dynamics on patterns of behavior, one that is usually easy to implement.

The behavior modes are characterized by an eigenvalue and the associated right and left eigenvectors. A simplified model that is good relative to a behavior mode will retain the eigenvalues and eigenvectors. Because $\underline{x}$ is of length N and $\tilde{x}$ is of length n, it is not possible to do this exactly. However, it may be possible to retain elements of the eigenvectors corresponding to components of $\underline{\tilde{x}}$. This allows for easy interpretation of the $\underline{\tilde{x}}$ vector as a subvector of $\underline{x}$.

Early work on this includes that of Davison (1966) and Marshall (1966) who looked at the problem of how to choose the $\underline{\tilde{A}}$ matrix to preserve certain apparently important modes. The analysis here does not go into any detail on the choice of the $\underline{x}$'s to include, but rather given that these have been chosen shows how to develop a simplified model retaining the right eigenvectors. More recent work in this area can be found in Gopal and Mehta (1982), Litz (1980) and Mahmoud and Singh (1982).

One of the key issues in the simplification process is the choice of the variables to be included in the simplified model. In particular, which variables are important in generating the behavior modes of interest. This is an important issue and was explicitly addressed in Perez (1981) and in Perez, Schweppe, and Verghese (1982a, 1982b). In these works the idea of a participation factor is introduced. The participation factor measures the importance of a state (or level) variable in generating a behavior mode. It takes into account both the amount that the variable moves over the course of a behavior mode and the amount that changes in the variable impact on the behavior mode.

Further discussion of which variables to include in a model is given in Forrester (1982, 1983). In this work the more technical means for selecting variables are considered in terms of the analysis of feedback loops. This allows for some blending with more traditional approaches in the Systems Dynamics field.

Along these lines further analysis is given in Eberlein (1984). This analysis suggests that a somewhat broader based approach to model simplification and considers explicitly such issues as the coupling of behavior modes through variables of interest. Again though, the emphasis is on determining which variables are important in generating given behavior modes.

The analysis in the literature discussed above is all strictly applicable only to the case of a linear dynamic model in the form of equation 1. In general, the models developed in system dynamics are nonlinear. This does not mean that the available results are inapplicable, but simply that there has to be a good deal of judgement as to their validity and usefulness. The methods for the analysis of nonlinear models are quite weak and there are few general results. In order to deal with the problem we concentrate on a very specific set of issues in nonlinearity, the concept of shifting loop dominance. Even so, the results developed can be no more than imperfect tools useful, but certainly not yielding any prepackaged results.

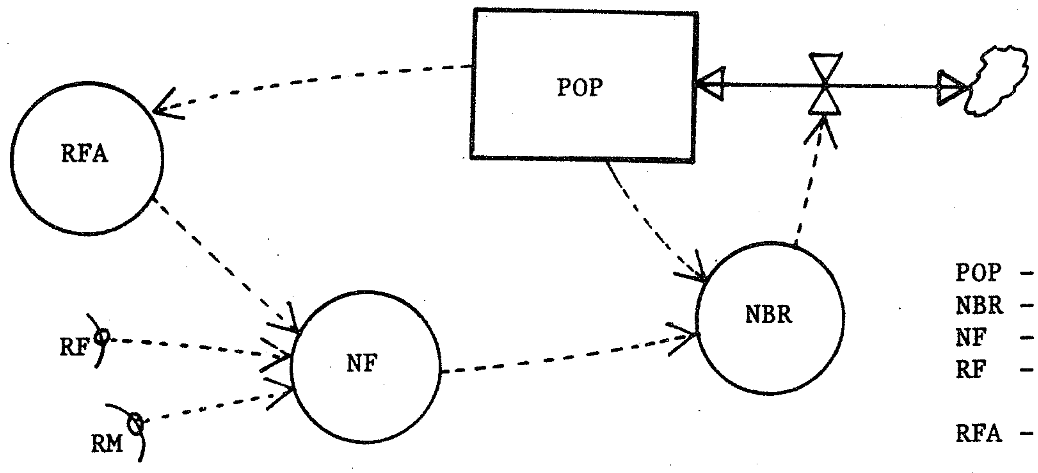
SHIFTING LOOP DOMINANCE

Shifting loop dominance is an important and well recognized phenomenon in dynamic systems. It is also an inherently nonlinear phenomenon. Basically, shifting loop dominance is the result of changes in gains around different loops. A common example of this is that of a positive loop that goes from growth to stabilization or decay. This will occur as the gain around the loop goes from larger than one to smaller than one. A very simple example of this is the population model shown in Figure 1. Initially, the relative food availability is high and, as a consequence, the net birth rate is also high. As drawn, the polarity of the loop is determined by the sign of the net birth rate, which starts out positive implying a positive loop. However, as the population increases relative food availability will fall, this causes the death rate to rise causing the net birth rate to fall and fall and eventually go to zero. As this occurs the negative loop through food availability is said to dominate.

The effect of the shifting loop dominance is not to remove the originally dominent loop, but rather to alter it so that its influence on the system changes. Essentially the negative loop is controlling the positive loop - and thereby the system behavior. This is different from controlling the system behavior directly since it suggests that there is something important to be learned by considering the changes over time in the positive loop.

The system shown in Figure 1 has only one level and it is therefore easy to determine the eigenvalue. This is given by (RF-RM*f(RFA)) where f is the function appearing in Figure 2. At low levels of relative food availability the death rate becomes very high and the eigenvalue will become negative. Starting the model with high relative food availability the loop through net birth rate is positive and generates exponential growth. As this happens, however, the food availability necessarily falls off and weakens this positive loop. At a sufficiently high population the positive loop will become a negative loop as the death rate rises above the birth rate. In this case the negative loop through food availability controls the polarity and dynamic consequences of the loop through net births.

It is important to note that the behavior in the model is still very much a consequence of the feedback through the net birth rate loop. The point is that the nature of this feedback has been changed by another loop. The distinction between



POP - Population
NBR - Net Birth Rate
NF - Net Fertility
RF - Reference Fertility
RFA - Relative Food Availability
RM - Reference Mortality

NF = RF - f(RFA)*RM

Figure 1 A simple Population Growth Model

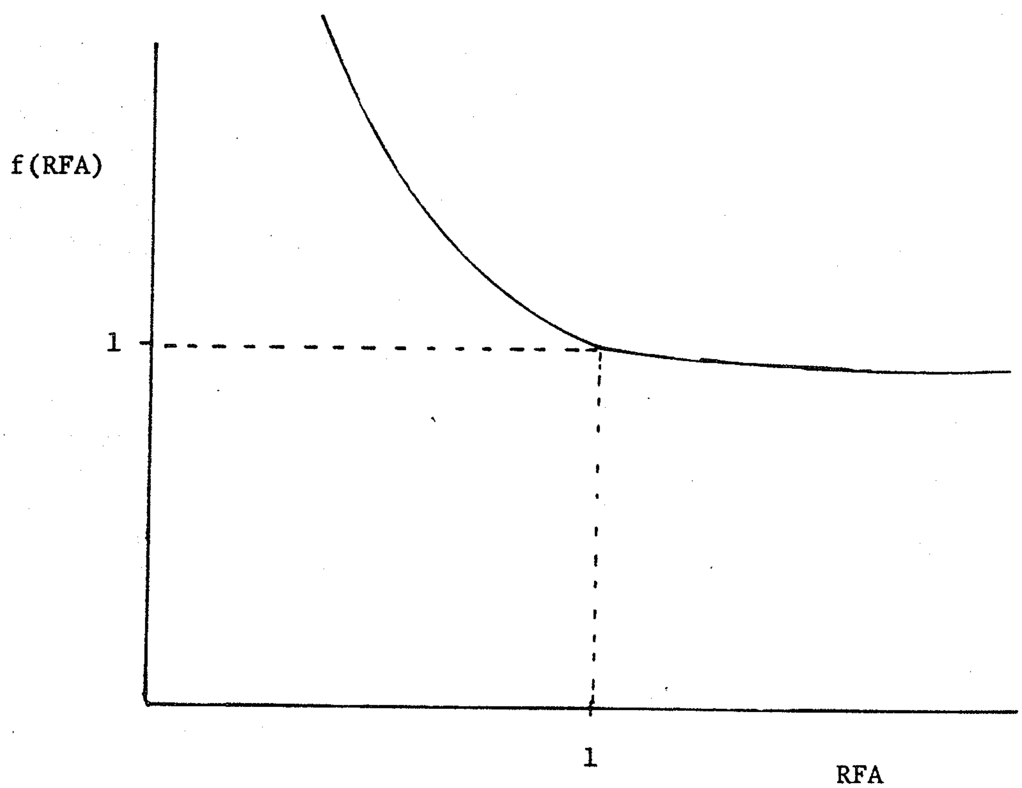


Figure 2 The Effect of Food Availability on Mortality

one loop altering another loop and that loop controlling the behavior of a model may at first glance seem arbitrary. In terms of using linear model analysis techniques on the model it is important. If we think of a loop as altering the eigenvalue resulting from another loop (or loops) then we can try to incorporate both the mechanisms generating the eigenvalue and the loop altering these in the simplified model.

In using linear analysis on a nonlinear model it is necessary to linearize the model. If the model is written as

$$\dot{\underline{x}} = \underline{g}(\underline{y}, \underline{u}, \underline{\varepsilon}) \tag{3}$$

with $\underline{g}$ a continuously differentiable function then the linearization of the model will be given by

$$\dot{\underline{x}} = \frac{\partial}{\partial \underline{x}} g(\underline{x}_0, \underline{u}_0, \underline{0})(\underline{x}-\underline{x}_0) + \frac{\partial}{\partial \underline{U}} g(\underline{x}_0, \underline{u}_0, \underline{0})(\underline{u}-\underline{u}_0) + \frac{\partial g}{\partial \underline{\varepsilon}} g(\underline{x}_0, \underline{u}_0, \underline{0})\underline{\varepsilon} \tag{4}$$

The eigenvalues will be those of $\frac{\partial}{\partial \underline{x}} g(x_0, U_0, 0)$ and will clearly vary over time as the derivative of $\underline{g}$ changes. However, as long as the derivative of $\underline{g}$ is continuous the change in the eigenvalues will be continuous over time. As a consequence it will be possible to trace the evolution of an eigenvalue associated with a particular feedback loop. This means that it is possible to trace the characteristics (growth, oscillation, decay, etc.) of a feedback loop over time.

The determination of which feedback loop is associated with which behavior mode is what the tools for linear model analysis are used for. In general this is not an exact correspondence, but only an approximate one. In addition, for different points in a simulation path the approximation may become worse. Stated alternatively, at different times different feedback loops may become important in the determination of a behavior mode. So that the approach we are taking is not by any means exact. It represents an heuristic and useful method for analyzing the determinants of behavior modes and using these to formulate simplified models.

LOCAL ANALYSIS

In order to analyze the problem we will make use of local analysis of the model. This is clearly a limitation on the tools we will develop but is necessary to make the issues tractable. The approach to this problem is relatively straightforward. It is assumed that a pattern of behavior such as growth turning into decay has been recognized and needs to be analyzed. In order to do this the growth behavior mode must first be recognized and the important loop or loops identified. It will be assumed that this has been done.

With the loops in hand we can talk of λ, the eigenvalue associated with the mode. The eigenvalue will depend upon the value of the state variables, that is we can write $\lambda(\underline{x})$ where $\underline{x}$ is the set of endogenous variables. The eigenvalue λ represents a mode of interest, for example a growth mode. The linear analysis gives us the tools for analyzing this mode. In addition, though, the eigenvalue changes over time as the state variables change. This is a nonlinear phenomenon and the question we now pose what states are important in changing λ.

There are two things required for a state to have an important influence on the eigenvalue of interest. First, a change in the value of the variable must change the eigenvalue, and second the variable must change value. The second of these sounds somewhat odd, but is very important. Clearly if a variable never changes then its inclusion in a simplified model, or as an element of an explanation of behavior, cannot be justified. More importantly though, because we are interested in internally generated dynamics the variable must be changed because of the influence of the behavior modes of interest.

This last point makes the determination of which variables to include in the simplified model difficult. There will in general be a variety of different feedback paths along which the variables important to an eigenvalue can influence another variable. We will restrict our attention to only the most direct path. That is, the movement of a variable over the course of the behavior mode of interest. The influence of a behavior mode on the i'th state variable is given by the corresponding component of the right eigenvector associated with the behavior mode.

The influence of the state variable on the eigenvalue can be measured in terms of the derivative of the eigenvalue with respect to the state variable. That is, we consider $\frac{\partial}{\partial \underline{x}} \lambda(x)$.

If we let $\underline{A}$ denote the linearized dynamics matrix then we can write the above derivative in terms of the element of $\underline{A}$ as

$$\frac{\partial}{\partial \underline{x}} \lambda(\underline{x}) = \sum_i \sum_j \frac{\partial \lambda}{\partial q_{ij}} \frac{\partial a_{ij}}{\partial \underline{x}} \qquad (5)$$

If we assume that the eigenvectors are normalized to have inner product one then $\frac{\partial \lambda}{\partial g_{ij}} = \ell_i r_j$ with $\underline{\ell}$ and $\underline{r}$ the left and right eigenvalues associated with λ. Using

this we can rewrite equation 5 as

$$\frac{\partial \lambda}{\partial \underline{x}} = \sum_i \sum_j \ell_i r_j \frac{\partial g_{ij}}{\partial \underline{x}} . \qquad (6)$$

To incorporate the influence of the behavior mode on the state we combine the derivative of the eigenvalue with the corresponding component of the right eigenvector. Writing things in matrix notation, with T denoting transpose, this gives the mode sensitivity measure

$$MS(x_k) = \underline{\ell}^T \frac{\partial \underline{A}}{\partial x_k} \underline{r}\, r_k . \qquad (7)$$

Note that the mode sensitivity will be high when the mode does influence the state variable substantially and the state variable also influences the mode. In other cases either r_k or the derivative of the eigenvalue will be small and so, therefore, will be MS.

Relative mode sensitivities will not be influenced by the units of measurement of different state variables. That is, the ratio of the mode sensitivities with respect to the ith and jth variables is independent of the units of measurement. To see this note that changing the units of measurement of say x_1 from dollars to cents will decrease the derivative of the eigenvalue by a factor of 100. At the same time increasing the first component of the eigenvector by 100 will give an eigenvector consistent with the newly measured state variable. These two changes would suggest that there would be no change in the mode sensitivity. However, the normalization for the right eigenvector is arbitrary and changing the normalization will change all mode sensitivities proportionally.

It would, of course, be nice if there were a clearly interpretable absolute measure of mode sensitivity. Unfortunately there is none as this would require a precise definition of the degree of nonlinearity in the model. For a linear model the modal sensitivities will all be 0. And if it is felt that the model is nearly linear it is unlikely to be appropriate to make use of the modal sensitivities, or to use anything but the linear model. The determination of the degree of nonlinearity in a model is a topic beyond the scope of this paper. However, if the mode sensitivities are normalized to have their absolute values sum to one there will be a conveniently usable measure.

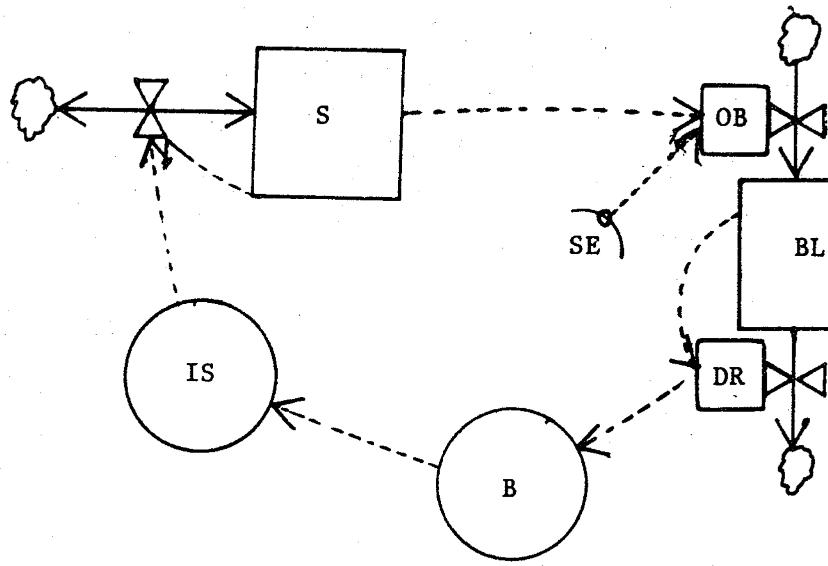
The variables with the high mode sensitivities have to be included in the simplified model. These variables may include some variables already in the simplified model as well as others that are not. In either case it should be recognized that the mode sensitivities indicate that a nonlinear relationship incorporating the variable is called for. The exact nature of this nonlinearity cannot be determined from what has been stated, but it should follow the original model structure. The nonlinearity incorporated should also be such that, in the simplified model, the derivative of the eigenvalue of interest with respect to the included variable should be close to, or match, that in the original model.

APPLICATION TO THE MARKET GROWTH MODEL

In order to get some feeling for how useful this approach may be we have applied it to the market growth model of Forrester (1968). In this model Forrester considers the problem of market growth in a limitless market which responds only to the level of the sales force and the delivery delay for the product. Forrester identified a basic positive loop through the sales force, sales, and the budget for sales responsible for growth (see Figure 3). Working against this is negative loop that decreases the effectiveness of the salesforce as the delivery delay rises. An example, of course, of shifting loop dominance.

We have rewritten the model, slightly changing the table functions for analytic convenience. We linearized the model at the initial values, and along the simulation path for the first year. During this time there is one growth root, implying a rate of growth initially of close to 4% per month. Over the first year of the simulation this falls slowly to imply a growth rate of about 3% per month at the end of the first year. As time progresses, the root combines with another to form a complex root, which implies growth and oscillation of a very long duration.

Using the technique of linear model analysis the basic elements of the growth loop are clearly identified as backlog and the sales force. This is consistent with what is shown in Figure 3. Because of the way the calculations are done only the level variables are included when the elements required for the simplified model are determined. It is necessary to incorporate additional variables to make the simple model easily understood. Given a simplified model containing the salesforce and backlog elements we want to ask what causes the decreasing rate of growth.



B - Budget
BL - Backlog
DR - Delivery Rate
IS - Indicated Salesmen
OB - Orders Booked
S - Salesmen
SE - Sales Effectiveness

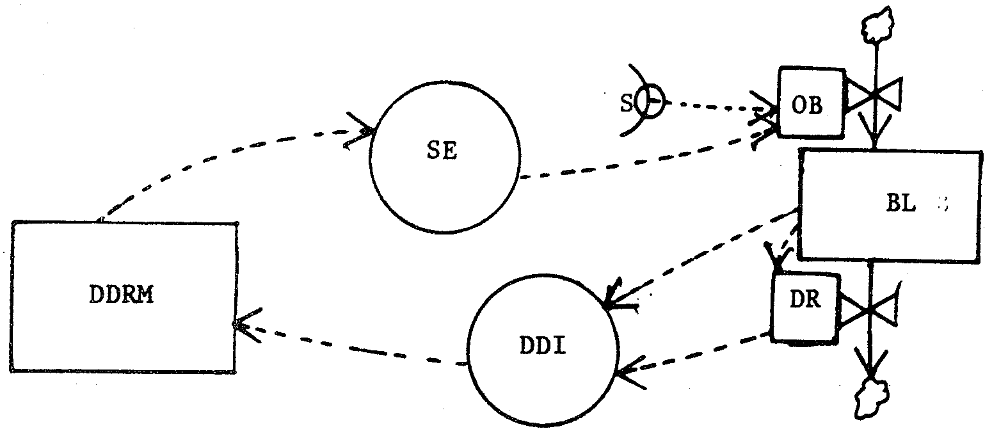
Figure 3 The Positive Loop Causing Growth

The second derivatives of the dynamics matrix were calculated at the beginning of the simulation. When these were weighted according to the formula given in equation 7 the results obtained are reported in Figure 4. The mode sensitivities have been normalized so that the absolute values add to one. Note that the sign of the derivatives have been maintained.

Mode sensitivity of growth mode with respect to	
Delivery Delay Recognized by Market	-.29
Salesforce	-.24
Backlog	-.24
Delivery Rate Average	-.14
Delivery Delay Traditional	-.06
Production Capacity	.04
Delivery Delay Recognized by Company	.03

Figure 4 Mode Sensibilities

From this it appears that the delivery delay recognized by the market is the important influence on the eigenvalue. This is consistent with the incorporation of a negative loop as shown in Figure 5 and certainly goes along with intuition.



BL - Backlog
DDI - Delivery Delay Indicated
DDRM - Delivery Delay Recognized by Market
DR - Delivery Rate
OB - Orders Booked
S - Salesmen
SE - Sales Effectiveness

Figure 5 The Negative Loop Retarding Growth

Following these guidelines we construct a simplified model that contains the salesforce, backlog and delivery delay recognized by the market as its levels. The simplified model has production capacity constant and removes some of the structure that does not seem to be central to the behavior mode under investigation. A dynamo equation listing of the simplified model is given in the appendix. The model is not, in itself, of great interest but serves to illustrate how an understanding of the basic mechanisms generating a pattern of behavior can be formalized.

The analysis of the market growth model has not offered any new insights into this model. The model is itself simple enough and comes complete with sufficient switches to generate the simplified model we present. What the analysis does show, however, is that the tools that we are using do yield results that are consistent with our previous understanding of this model. This is important, since that is a necessary condition for their usefulness in a more general setting.

The simplified model as we have developed it loses a lot of what is of interest in the market growth model. The expansion of production capacity and the effect of declining goals on the overall growth are obviously important issues, if not the key issues the model addresses. This shortcoming of the simplified model makes an important point. Simplified models are not, by their nature, as interesting and the models from which they derive. Richness and often realism are sacrificed for a simpler modeling aimed at a specific pattern of behavior. If this is not the only pattern of behavior of interest, then something will be missed. Simplification does not represent a magic tool by which understanding can be obtained. It can, however, be very useful when combined with hard work and other techniques for analyzing models.

NECESSARY EXTENSIONS

The approach discussed in this paper is highly experimental and has not yet been applied to any large models. There are two major barriers to its application. One is the lack of of suitable software for calculation of the mode sensitivities. The other is the great deal of computation required. Taking the derivative of the A matrix requires that N^3 derivatives be calculated. Even in our simple example this would involve 1000 calculations (the model having 10 states). While it is true that, because of the sparse nature of the A matrix, it will not be necessary to calculate this many derivatives, the burden it still substantial. In our example there were 26 nonzero entries in the A matrix, thus requiring 260 derivatives. While this computational burden is not overwhelming it is severe, and it is likely that some decrease in the required computation is possible. This is an area requiring further investigation.

There is another issue in this discussion and that is the consideration of only the level variables. While it is possible to explicitly allow for the inclusion of auxiliary variables in calculating the different sensitivities (see for example CCREMS 1983, Eberlein 1984, chapter 5) the use that can be made of these is not clear. This is a criticism of the linear model simplification techniques as well as the ones discussed in this paper. Because the auxilitary variables are an important part of any model their explicit inclusion would be helpful, and it would certainly make the construction of the simplified model easier.

CONCLUSIONS

The techniques of linear model analysis have often been applied to nonlinear models. Because the techniques assume complete linearity their application requires some caution and is not likely to succeed in the face of severe nonlinearities. By concentrating on a particular type of nonlinear phenomenon, specifically shifting loop dominance, it is possible to explicitly recognize the nonlinearity in developing the simplified model. Since developing a simplified model is the same as gaining an understanding from an existing model the tools develoepd are potentially very useful in analyzing complex nonlinear models. The application of the techniques yield results that are useful in analyzing models.

Though the results quoted in this paper are of a tentative nature there does seem to be some potential for the development of tools to analyze nonlinear models. The techniques discussed in this paper require further work before they can be easily implemented. In addition it is desirable to develop techniques to deal with other types of nonlinearities. If a collection of such techniques can be brought together in a unified package for model analysis the potential for increased model understanding and higher efficiency in model analysis is great.

APPENDIX: SIMPLIFIED MODEL EQUATION LISTING

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* SIMPLIFIED MARKET GROWTH MODEL
NOTE
NOTE THIS MODEL USES THE MNEMONICS IN FORRESTER (1268) AND
NOTE MAINTAINS AS FAR AS POSSIBLE THE MODEL STRUCTURE THEREIN
NOTE
NOTE FIRST THE POSITIVE SALES HIRING LOOP IDENTFIED B THE LINEAR
NOTE ANALYSIS AS MOST CLOSELY ASSOCIATED WITH THE GROWTH ROOT.
NOTE
L S.K=S.J+DT*SH.J                 SALESFORCE (PEOPLE)
N S=10
A SH.K=(IS.K-S.K)/SAT             SALESFORCE HIRING (PEOPLE/MONTH)
C SAT=20                          SALESFORCE ADJUSTMENT TIME (MONTHS)
A IS.K=B.K/SS                     INDICATED SALESFORCE (PEOPLE)
C SS=2000                         SALESFORCE SALARY ($/MONTH/PERSON)
A B.K=DR.K*RS                     BUDGET FOR SALESFORCE ($/MONTH)
NOTE
NOTE REVENUE FROM SALES HAS BEEN CHANGED TO MAINTAIN THE ROOT OF
NOTE INTEREST.
C RS=11.77                        REVENUE FROM SALES ($/UNIT)
A DR.K=PC*PCF.K                   DELIVERY RATE (UNITS/MONTH)
A PCF.K=TABHL(TPCF,DDM.K,0,5,.5) PRODUCTION CAPACITY FRACTION
T TPCF=0./.31/.45/.55/.63/.71/.77/.84/.89/.95/1      (DIMENSIONLESS
A DDM.K=BL.K/PC                   DELIVERY DELAY MINIMUM
L BL.K=BL.J+DT*(OB.J-DR.J)        BACKLOG (UNITS)
N BL=8000
A OB.K=S.K*SE.K
NOTE SALES EFFECTIVENESS AND THE EFFECT OF DELIVERY DELAY ON THE
NOTE MARKET.  IDENTIFIED AS THE CHIEF CAUSE OF CHANGE IN THE
NOTE GROWTH ROOT
A SE.K=SEDM.K*SEM                   SALES EFFECTIVENESS
NOTE                                        (UNITS/MONTH/PERSON)
C SEM=400                           SALES EFFECTIVENESS MAXIMUM
NOTE                                        (UNITS/MONTH/PERSON)
A SEDM.K=TABHL(TSEDM,DDRM.K,0,10,1) SALES EFFECTIVENESS FROM DELAY
NOTE                                     MULTIPLIER (DIMENSIONLESS)
T TSEDM=1/.97/.91/.87/.75/.65/.54/.41/.28/.15/.02
NOTE
L DDRM.K=DDRM.J+(DT/TDDRM)*(DDI.J-DDRM.J) DELIVERY DELAY RECOGNIZED
NOTE                                  BY MARKET (MONTHS)
N DDRM=DDI
C TDDRM=10                          TIME FOR DELIVERY DELAY
NOTE                                  RECOGIZED BY MARKET (MONTHS)
A DDI.K=BL.K/DR.K                   DELIVERY DELAY INDICATED (MONTHS)
C PC=12000                          PRODUCTION CAPACITY (UNITS/MONTH)
NOTE
NOTE CONTROL CARDS
NOTE
SPEC DT=.5/LENGTH=100/PRTPER=0/PLTPER=3
PLOT S/BL/DDRM
RUN
```

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