Period Doubling and Approach to Chaos in a Simple Context

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ABSTRACT

Studies of deterministic systems which exhibit apparently chaotic behavior are attracting much interest in disciplines ranging from physics to economics. A particularly interesting case of a simple electrical network has been studied recently in the physics literature with the objective of isolating minimal characteristics essential to chaotic behavior. A system dynamics formulation has been given to the numerical simulation of this system. Instructional laboratory exercises comprising both observations on the electrical circuit and computer simulation of the circuit are being implemented for upper level undergraduate and graduate students.

INTRODUCTION

In recent years much interest has developed in apparently chaotic behavior that occurs in deterministic systems, i.e., in systems governed by laws which include no explicitly random element. Systems from diverse disciplines have been studied, including physics (Eckmann, 1981; Ott, 1981) and economics (Day, 1983). Many systems that have been studied fall into one or the other of two categories. In one category are systems which are physically simple but not very transparent from an analytical point of view, e.g., a dripping water faucet. In the other category are systems which obey a simple analytical law but which do not correspond very clearly to a physically realizable system, e.g., the by now classic logistic equation $x'=4Kx(1-x)$.

It is of particular interest then to note recent reports of chaotic behavior in a simple electrical network which is readily observable in the laboratory and which is governed by equations that are analytically solvable, in one configuration of the system, by elementary means (Rollins, 1982). Moreover, one aim of that work has been to determine the minimum essential physical attributes of the system which lead to the chaotic behavior. Recognition of the similarity in the behaviors of electrical networks and systems from diverse disciplines suggests that study of chaotic behavior in a simple electrical circuit promises to be of quite general appeal.

The work which is reported here was undertaken with two rather different objectives in mind, one of a pedagogical nature and one related to possible system dynamics implications that might be inferred from this simple electrical system. A laboratory realization of the electrical network was desired in terms of apparatus available to an undergraduate instructional facility and which would yield experiments suitable as exercises for junior-senior-graduate level physics majors. It was also desired that a computer simulation be available for use by the students. The computer simulation should be relatively straightforward in use and should yield results that are at least qualitatively similar to phenomena observed in the laboratory work. The intent of the instructional exercises is not to present a rigorous treatment of the area, which is mathematically sophisticated, but to provide some familiarity with phenomena and concepts. Although these experiments and computer simulations can be of interest to disciplines other than physics, the import of the work would be largely lost on students who do not have a modest degree of sophistication about electrical circuits.

The second objective is more general in nature. In view of the fact that the features of the system which are essential to chaotic behavior can be identified explicitly, it is interesting to examine those features in the system dynamics formulation. This should facilitate identification of analogous features in other system dynamics problems, including socio-economic systems. However, exploration of system dynamics ramifications of the work is not related to the immediate departmental educational objectives and has not been pursued much as yet.

ELECTRICAL CIRCUIT

The electrical circuit that demonstrates chaotic behavior, under suitable parameter conditions, is depicted in Fig. 1. The circuit is a series combination of driving voltage, diode, inductance, and resistance. The network comprising V, R', and C represents the diode. The resistance R may be comprised of three parts: a resistor inserted into the circuit, the resistance associated with the inductor, and the output resistance of the driving voltage circuit. The reactance of the inductance, at the driving frequency, should be much larger than the resistance.

The diode will alternately find itself in conducting and non-conducting states. For analytical purposes, the diode may be represented by an equivalent circuit. When conducting, the diode is replaced by an e.m.f. (battery) of voltage V. The polarity of the e.m.f. is such that it opposes the forward current flow. This e.m.f. simulates the voltage drop across a forward biased diode, about 0.6 volt for a forward biased p-n junction in silicon. When in the non-conducting state, the diode may be replaced by a capacitor. With reference to Fig. 1, when the diode is in the conducting state, the resistance R' is considered negligible. When the diode is in the non-conducting state, the resistance R' is very large compared with the impedance of the capacitor C at the driving frequency. When the diode is non-conducting, the circuit is a driven R-L-C circuit. The resonance frequency for a series R-L-C circuit occurs when the reactances of the inductance and capacitance are equal. The investigations here are limited to driving frequencies that are within about twenty per cent of the resonance frequency.

Two categories of diodes may be employed in order to investigate the onset of chaos. In one category are diodes in which the capacitance, under reverse bias, is considered to be independent of voltage across the diode. The capacitance in the real diode will always depend to some extent upon the voltage, but this dependence is neglected in the circuit analysis. In the other category are tuning diodes, or varactors, in which the capacitance depends strongly on voltage (Linsay, 1981; Testa, 1982). Analytic solution of the circuit equations is possible only for the first category. The instructional work,

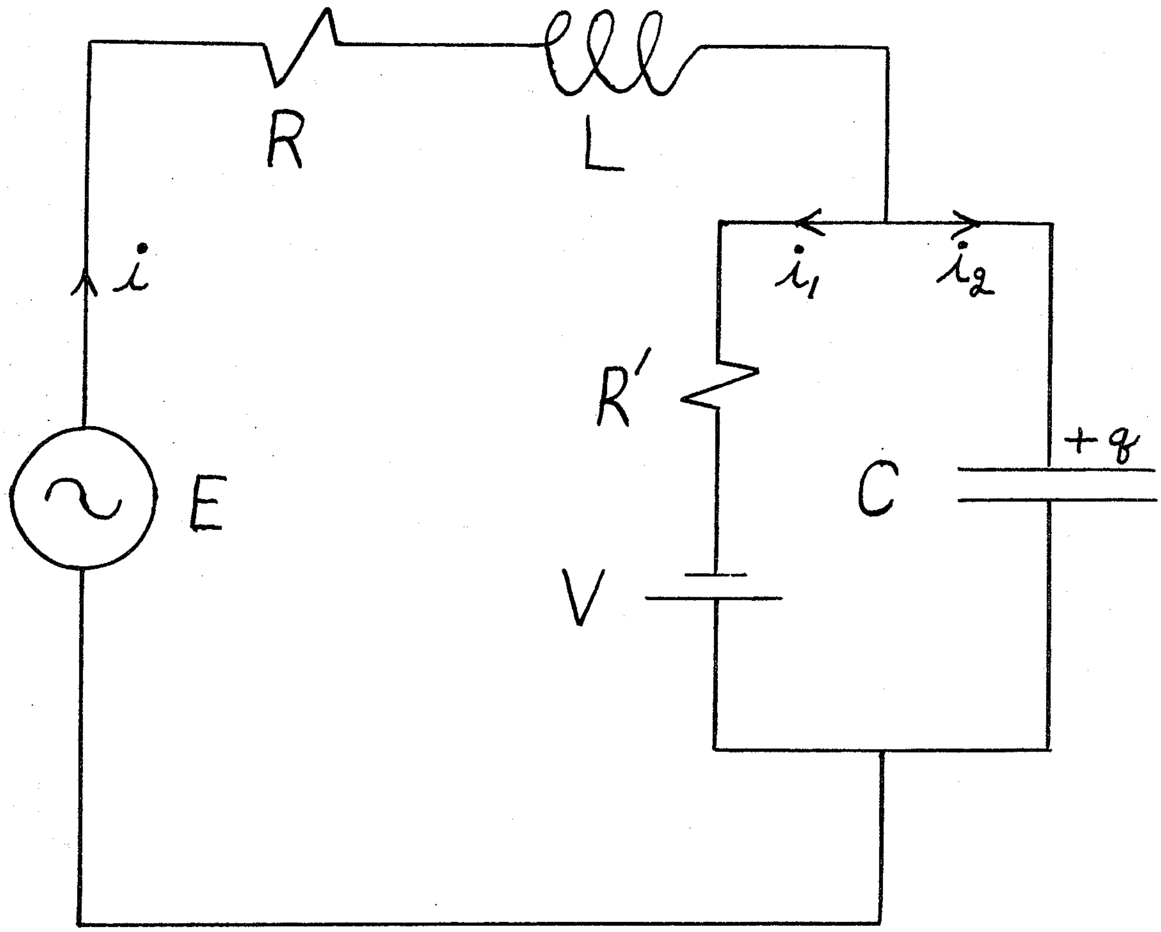

Fig. 1. The electrical circuit: series combination of driving voltage (E), diode (V, R', C), inductance (L), and resistance (R).

except perhaps for a subsidiary experimental demonstration, will be limited to the former category of diodes.

The transition of the diode from the conducting state to the non-conducting state does not occur just as the current through the diode changes direction. With reference to Fig. 1, the current is positive (+) when the diode is non-conducting, and the current flows in the sense of the arrow. The current is negative (-) when the diode is conducting, and the current flows in the sense opposite to the arrow. Change in the state of the diode from conducting to non-conducting does not occur just as the current changes from (-) to (+). The transition to the non-conducting state is delayed by the "reverse recovery time" following the change in direction of current from (-) to (+) (Millman, 1965, pp. 749-752). Computer simulations reported here assume the reverse recovery time is determined by the maximum magnitude of the diode current during the period of forward current flow immediately preceding the transition (Rollins, 1982). Recent work suggests a more complicated dependence of reverse recovery time on past history of the circuit in order to secure quantitative agreement between experiments and computer simulations (Hunt, 1984).

In an ideal diode, the forward bias voltage, V, would be zero, and the reverse recovery time would be zero. Departures from both of these idealized conditions are necessary in order to secure chaotic behavior. One might suspect that the onset of chaotic behavior is associated with non-linearity in the circuit due to a voltage dependent capacitance for the reverse-biased diode. It would be difficult to rule against this interpretation on the basis of experimental work because of the inevitable dependence of capacitance on voltage in a real diode. However, chaotic behavior was found in computer simulations with constant capacitance under reverse bias (Rollins, 1982).

The occurrence of chaotic behavior and the behavior patterns that characterize the onset of chaos are very sensitive functions of diode properties. The reverse recovery time must be of the order of the period of the driving voltage, which in turn is about equal to the period of natural oscillations of the R-L-C circuit under reverse bias. In the present work, frequencies in the range of several hundred thousand hertz up to about one megahertz have been used. No chaotic behavior is noted for fast signal diodes. Power diodes and some tuning diodes show chaotic behavior, but the latter have voltage dependent capacitance under reverse bias. Hence, certain power diodes are deemed most suitable for the instructional work.

There are additional practical considerations related to operation of the circuit in an instructional laboratory. The oscillator or signal generator that furnishes the driving voltage should have a low output impedance. Otherwise, loading of the generator by the circuit, which is considerable for large amplitude driving voltages and near resonance frequencies, will cause complications in circuit behavior. An impedance matching circuit capable of handling driving voltage amplitudes in the 10-20 volt range may have to be provided to go between the signal generator and the circuit.

It is necessary to monitor certain voltages or the current in the circuit. Measurements in the instructional laboratory will be limited to monitoring the current and the driving voltage. The former is monitored by observing the voltage drop across a resistor inserted in the circuit, e.g., R in Fig. 1. The two voltages will be observed on the channels of a dual channel oscilloscope. Instant photographs of the screen may be made in order to have a permanent record of the observations. This procedure is satisfactory for the observation of waveforms with periodicities up to perhaps sixteen times the period of the driving voltage. Higher periodicities can be found in the computer simulations and might be expected in the laboratory.

In the development work for the laboratory experiments and in research reports in the literature, additional types of observations have been made, which are deemed less suitable for the instructional laboratory. It is interesting to know the voltage across the diode. However, the impedance across the diode is very large, and any system attached across the diode for observational purposes is likely to alter the circuit in a significant manner. Thus, the circuit of Fig. 1 should be replaced by a more complicated circuit, which must include the input circuit to the test instrument and the connecting cable. Although interesting effects will be observed, the representation of the circuit for analytical purposes may become obscure. Similar remarks apply to the making of observations of the voltage across the inductor.

Most studies in the research literature have employed a spectrum analyzer in order to monitor operation of the circuit. The analyzer determines the frequency components present in, for example, the current and displays peaks representative of the relative amplitudes of the various frequency components. Such an instrument greatly facilitates identification of the frequency structure of the system response, especially for the higher periodicities. A spectrum analyzer has been employed in the development work but is not planned for inclusion in the instructional laboratory. The controlling consideration is that the analyzer is expensive, and no such instrument is available, except on a temporary loan basis, for instructional usage. Even if such an instrument were available, observations with the oscilloscope would be stressed because they provide a more literal representation of circuit operation. However, instructional laboratory work should be devoted, in part, to familiarization of the students with state-of-the-art instrumentation capabilities. Hence, inclusion of spectrum analyzer observations would be a worthwhile supplement.

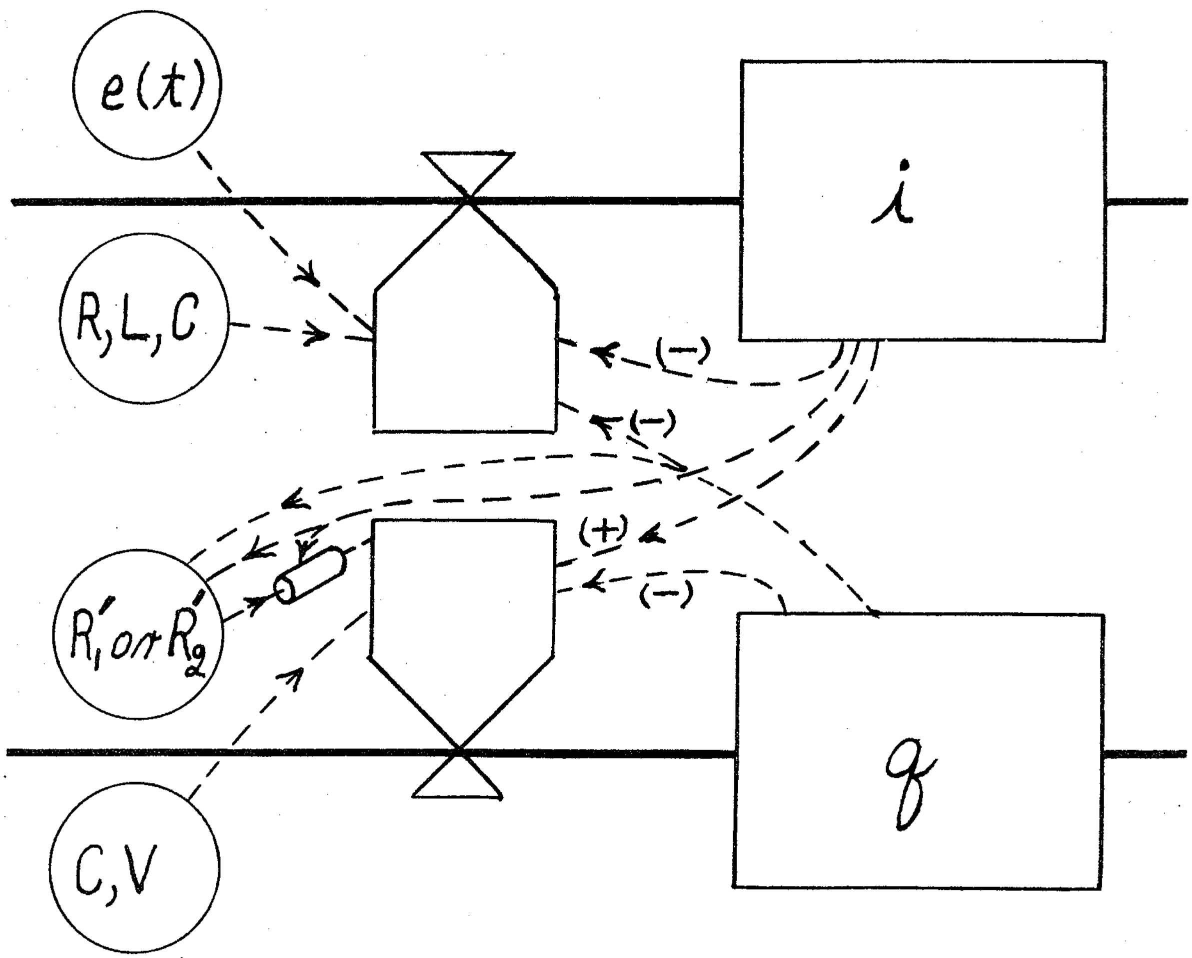

Fig. 2. Feedback structure of the system dynamics problem.

SYSTEM DYNAMICS FORMULATION

The circuit of Fig. 1 may be described by a set of two first-order, coupled differential equations

$$di/dt = -Ri/L - q/LC + eL \tag{1}$$

$$dq/dt = i - q/R'C - V/R' \quad . \tag{2}$$

Here, e is the time-dependent driving voltage of amplitude E, and q is the charge on the capacitor. The symbols in these equations are proportional to the corresponding physical quantities when measured in conventional units but are not equal to the physical quantities. In order to eliminate superfluous variables and avoid very large or very small numbers, which is advantageous for computer analysis, all quantities have been scaled in terms of "natural" units, defined in such a way as to be appropriate to this particular problem. For example, times are measured in terms of a unit of time that is about equal to the period of natural oscillations of the series R-L-C circuit.

The feedback loops implicit in the equations may be visualized in terms of the system structure depicted in Fig. 2. The diagram has been drawn for the analytically solvable case in which the capacitance C does not depend upon the voltage across the capacitor. However, inclusion of this complication would not upset solution of the equations in a conventional system dynamics numerical integration of the equations.

A property of the system that is essential to chaotic behavior is incorporated into the R_1'-R_2' part of the diagram. R_1' and R_2' denote the two values that the resistor R' of Fig. 1 may assume in order to simulate conducting and non-conducting states of the diode. A time delay in the transmission of this information to the system, dependent upon the current, is indicated, and there will be no chaotic behavior if this delay time is too small. No chaotic behavior occurs if the parameter V, the diode forward bias, becomes too small.

COMPUTER SIMULATION

The conventional system dynamics procedure for computer simulation of the circuit is to numerically integrate the differential equations (1) and (2). However, advantage may be taken of the fact that the equations can be solved analytically, provided the capacitance is assumed independent of the voltage across the capacitance, i.e., independent of q. An alternative way of writing the circuit equatioons, the more customary way, is

$$di/dt + iR = V + e \qquad \text{(diode conducting)} \tag{3}$$

$$d^2Q/dt^2 + R\, dQ/dt + Q = e \qquad \text{(diode non-conducting)} \quad . \tag{4}$$

The Q here is not the q of Fig. 1, but is related to i according to

$$i = dQ/dt \quad . \tag{5}$$

The driving voltage is

$$e = E \cos(\Omega t + G) \quad . \tag{6}$$

E is the amplitude of the driving voltage, Ω is the angular frequency of the driving voltage, and G is a phase angle. In the natural units employed here, the natural frequency of the R-L-C circuit is unity to a sufficient approximation.

The intrinsic non-linearity of the system occurs in switching between the two differential equations. For an ideal diode, transition from conducting to non-conducting state occurs as the current goes from (-) to (+). In the practical diode, the need to sweep out minority current carriers near the p-n junction causes a delay that is assumed to obey (Rollins, 1982)

$$t_d = t_{do} \, (1 - \exp(-I_m/I_c)) \quad . \tag{7}$$

The magnitude of the largest forward current that occurred prior to the transition is denoted I_m. I_c and t_{do} denote empirical parameters that depend on the particular diode. For an ideal diode, transition from non-conducting to conducting state occurs when the voltage drop across the diode passes through zero. For the practical diode, a forward bias is required, and transition is delayed until the voltage across the capacitance C reaches the value V, with polarity as indicated by the battery in Fig. 1.

It proves to be a tedious business to search for interesting behavior patterns. The computer program utilized for that purpose carries through the algebra associated with the analytic solutions of the differential equations (3) and (4), switches between equations in accordance with time lag (7) and bias voltage V, and matches solutions smoothly upon switching. Some special features were included in the program to accommodate several categories of behavior patterns and to facilitate running through many diode on-off cycles in order to establish the periodicity, if any, of a pattern. Although this procedure for finding interesting behavior patterns was deemed more convenient than the more conventional system dynamics procedure, the results are equivalent to direct use of equations (1) and (2), which are more obviously connected with the feedback structure indicated in Fig. 2.

COMPUTER SIMULATION RESULTS

Behavior patterns of the circuit are sensitive functions of the several parameters that are available, which include: amplitude of driving voltage (E), two parameters that specify delay time in switching from the conducting to the non-conducting state (t_{do} and I_c), the forward bias voltage of the diode (V), the resistance in the circuit (R), and the frequency of the driving voltage (Ω). The absence of some parameters that one might expect, e.g., the inductance and the capacitance, is due to the scaling of parameters in natural units, which essentially converts quantities into dimensionless ratios and reduces the number of independent variables to a minimum.

A myriad of behavior patterns can be produced by choice of the several parameters. Limited investigations have been carried out for variation in some parameters, e.g., R and Ω. More extensive investigations have been carried out for variation in the amplitude of the driving voltage (E), the bias voltage (V), and the delay time as affected by t_{do}.

An interesting sequence of patterns may be observed as some parameter is monotonically changed. Table 1 provides a record of the nature of behavior patterns observed as the delay time parameter t_{do} is increased. Larger values of t_{do} correspond to longer delay times in switching from the conducting to the non-conducting state. The entry "period 5," for example, denotes that the response of the circuit is periodic with a period equal to five times the period of the driving voltage. The notation "unstable" denotes that no periodicity can be identified for the response of the system, and the state may be termed "chaotic." Perusal of the t_{do} values may convey some feeling for the seemingly erratic sequences observed and the sensitivity of the response patterns to parameter values. The sequence listed is not intended to imply the absence of other patterns for intermediate values of t_{do}. For example, the range t_{do} = 6.35 - 6.495 is denoted "unstable." However, there might be one or more "windows" of stability in this range that have not been detected. One such window that was detected occurs at t_{do} = 6.5, where a stable pattern of periodicity 16 was found. Roundoff in the numerical computations effectively introduces noise into the system, and it is well known that noise upsets the stability of a system, especially in a region of great parameter sensitivity. Thus, a marginally stable pattern may not appear in a computer simulation.

t_{do}	Period	t_{do}	Period
0 - 0.90	1	6.35 - 6.495	unstable
0.91 - 3.32	2	6.500	16
3.35 - 3.70	4	6.505 - 6.7	unstable
3.71	9	6.8	11
3.75	5	7.0	unstable
3.82	9	7.3	25
4.5	10	7.35	unstable
5.5	7	7.4 - 9	3
6.2	28	10	10
6.3	21	11	unstable

Table 1. Exemplary sequence of behavior patterns observed as the time lag parameter t_{do} is varied.

Unstable				Period 9
-8.7	-2.1	0	-11.4	0
0	-11.2	0	-1.2	-9.0
-7.4	-11.3	-2.5	-11.7	-1.4
-10.6	0	-8.9	-9.2	-10.8
-6.4	-11.7	-10.0	-9.3	-5.7
-6.6	-3.4	-9.7	-2.6	-8.8
-6.8	-10.6	-10.7	-9.8	-1.9
-3.3	-8.8	-6.2	0	-10.8
0	-11.1	-8.0	-11.3	repeats
-0.4	-10.5	-9.2	-9.3	

Table 2. Exemplary sequence of current values for an unstable case. In the last column is a corresponding sequence for a pattern of period 9.

An exemplary set of numbers that are monitored in the search for periodicities is listed in Table 2 for an unstable pattern. The numbers represent the current in the circuit at a particular juncture in the computational cycle. The zeros do not have any particular significance. The diode switching cycles fall into a few categories. When one particular category is recognized, the next computational cycle begins with current equal to zero. The essential point is that the pattern of numbers repeats if the circuit evolves into a periodic state. Some features of the "chaotic" patterns are suggested by the table. The numbers lie within some limited range, which seems to extend from 0 to perhaps about -12 for the case illustrated. All numbers within this range are not equally likely. Values around -10 occur frequently, while values around -5 have not appeared.

Detection of very long periods can be difficult. It is possible that such a period may be identified erroneously as unstable because the pattern of numbers within the long period tends to be similar to an unstable pattern, and one may simply not look long enough for periodicity. The problem is aggravated by long transient behavior. One does not know a priori what starting conditions to use for a computation, and the stable pattern may emerge only after many computational cycles. It is observed that in some regions of parameter space, approach to stability is very slow. In fact, even short periodicities can be missed if an unfortuitous choice of starting conditions coincides with a region of slow convergence.

In the last column of Table 2 is listed the sequence of current values found for a pattern of periodicity 9. (The periodicity is not to be inferred from counting the numbers, which total only eight in this instance.)

Typical plots of system response, current vs. time, are shown in Figs. 3 and 4. A sequence of period doublings, from period 1 (same as driving voltage) to period 2 and then to period 4, is shown as Fig. 3. A period 3 pattern, which is a frequently observed period both in computer simulations and in experiments, is shown in Fig. 4, along with a plot of the driving voltage. The phase relationship between driving voltage and current is a starting parameter whose correct value is not known a priori. Only after enough computational cycles have elapsed for the response to evolve into a stable pattern can the proper phase relationship, as depicted in the figure, be determined.

EXPERIMENTAL RESULTS

Experiments have been conducted using a variety of diodes. Chaotic behavior has been observed in power diodes that have long reverse recovery times and in tuning diodes that have voltage dependent capacitance under reverse bias, but not in signal diodes that have short reverse recovery times. The qualitative behavior of the circuits is generally similar to the results of computer simulations. An approach to chaos scenario of the period doubling type, i.e., periodicities 1, 2, 4, 8 ... , has been observed, along with de-bifurcations (periodicities ... 4, 2, 1) and other patterns, including periodicities of 3, 5, etc. Chaos regions interspersed with windows of stability have been observed.

For the most part, investigations in parameter space involve changes in the amplitude of the driving voltage, although effects of changing driving frequency or circuit resistance have been examined briefly. Some parameters are not amenable to clean experimental study. Changes in the switching delay characteristics require a change in diode type, and that effectively changes the circuit in other ways as well. The diode forward bias voltage is that appropriate to a silicon p-n junction in all cases.

No effort has been made as yet to match parameters in the computer simulations to a particular experimental circuit in order to secure quantitative agreement between the two studies. Photographs of several experimental response patterns are shown in Fig. 5.

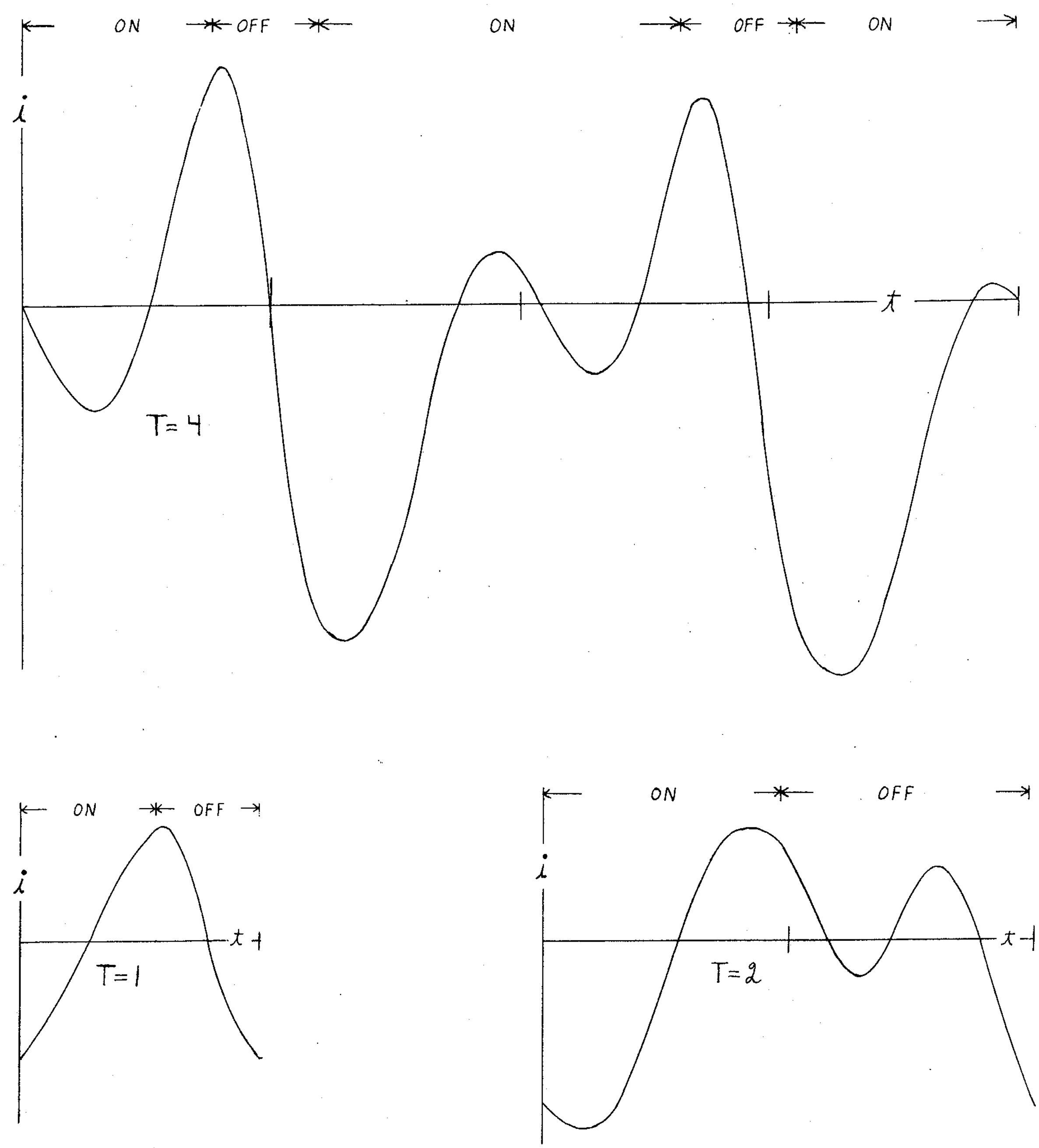

Fig. 3. Current vs. time for cases with periodicities 1, 2, and 4.

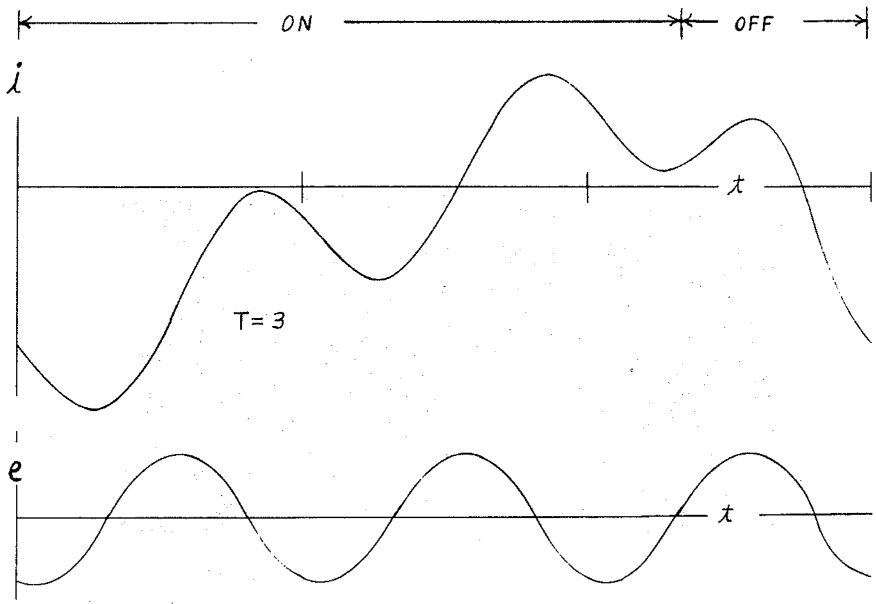

Fig. 4. Currrent vs. time for case with periodicity 3. Plot of driving voltage vs. time, with stable phase relationship relative to current, is also shown.

The term "universality" has come to denote certain quantitative features of the approach of a system to chaotic behavior that seem to be common to diverse systems, independent of specific system details. The "Feigenbaum number" characterizes the period doubling scenario that is found for many systems. In terms of the amplitude of the driving voltage, we may define

$$d_n = \frac{E_{n+1} - E_n}{E_{n+2} - E_{n+1}} \quad . \tag{8}$$

For n large, d_n converges to the Feigenbaum number, 4.6692.. (Feigenbaum, 1979). Here E_n denotes the amplitude of the driving voltage that just causes the period of the system response to change from 2^{n-1} to 2^n, measured in multiples of the period of the driving voltage. Thus, when E reaches the value E_3, the period changes from 4 to 8. Often, convergence of d_n is rapid and a good approximation to the Feigenbaum number may be found for low values of n. A rough experimental determination has been made for a period doubling sequence which demonstrated periodicities 1, 2, 4, 8, and 16. The values found were $d_2 = 3.9$ and $d_3 = 4.5$.

CONCLUDING REMARKS

Both experiments and computer simulations amply demonstrate the rich variety of behavior patterns and sequences of patterns that can occur for a relatively simple, deterministic system. In some regions of parameter space, arbitrary starting conditions quickly lead to stable patterns of low periodicity. In other regions, long-lived transients ensue which eventually lead to complex patterns that are stable but of very long periodicity. Finally, unstable or

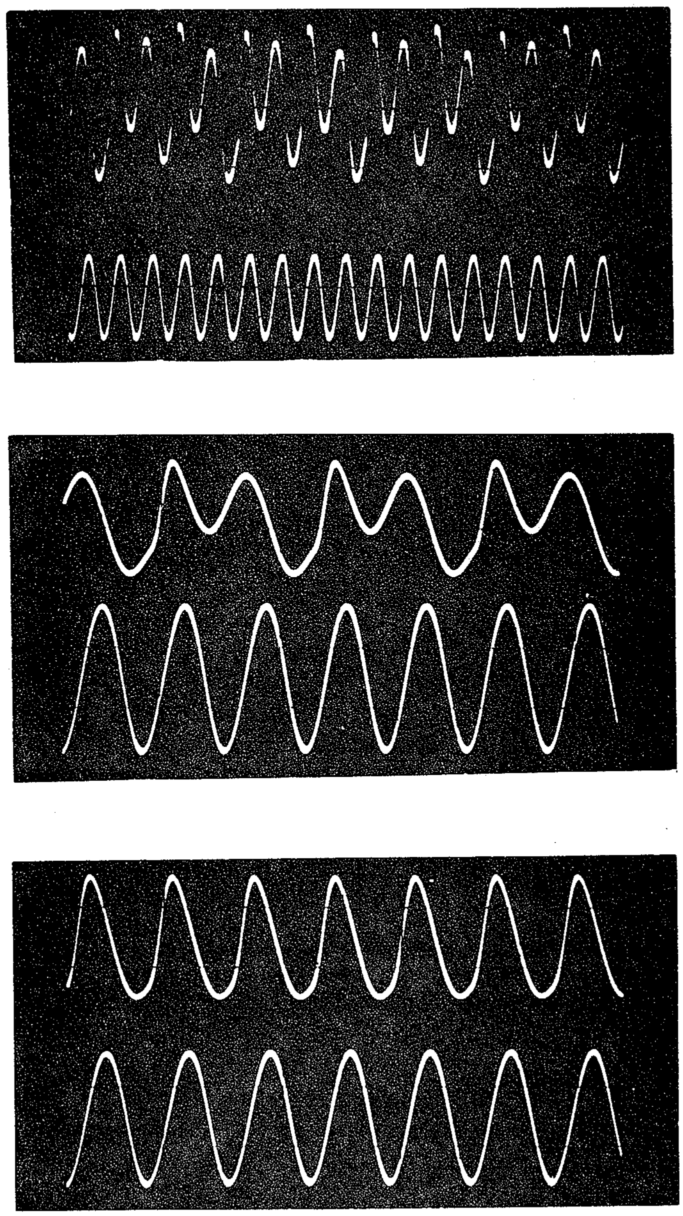

Fig. 5. Photographs of oscilloscope displays of current vs. time for experimental response patterns of periodicities 1, 2, and 4. The lower trace in each photograph shows the driving voltage.

chaotic behavior occurs in certain regions of parameter space.

A loose characterization of the elements essential to chaotic behavior in the system studied here includes the following. There are two distinct states in which the system may reside (conducting and non-conducting states of diode). Behavior within one state may be described by linear equations, but the transition between states introduces non-linearities. There is a bias in the system such that a system variable must "overshoot" its idealized transition value before the transition can occur (forward diode bias voltage). There is a time delay in the system such that the reverse transition occurs later than it would in an ideal system (diode reverse recovery time).

In regard to the time delay, the results of one series of computer simulations are suggestive, albeit certainly not conclusive. Simulations were made with a fixed time delay, i.e., a time delay that did not depend on any system variable. (In subsequent simulations, the delay depended on the diode current.) A wide variety of behavior patterns was observed, including long periodicities. However, in regions of parameter space similar to those employed later, no clear case of chaotic behavior was noted.

The richness of possible results, both in computer simulations and in experiments, has both advantageous and disadvantageous implications for instructional applications. On the one hand, student interest should be enhanced by the potential to investigate interesting situations where neither the student nor the professor can predict the outcomes. On the other hand, endless time can be devoted to aimless probings in parameter space without coming upon some of the important behavior patterns.

Tentative plans call for a series of projects. One project, preferably the first in the series, will be computer simulations of classic problems, e.g., the logistic equation, but with stipulated procedures for the most part. Several phenomena that are common to a broad range of systems will be encountered in the simplest possible mathematical context, including the period doubling scenario and the Feigenbaum number. Experience can also be gained with additional concepts that have not been described here. A rescaling parameter, usually denoted as α, describes the spacing of elements in the patterns that develop during the sequence of bifurcations (Feigenbaum, 1979; Schuster, 1984, pp. 33, 35). The rate of convergence from arbitrary starting conditions towards a stable pattern or, in a chaotic region, the rate of divergence from initially nearby starting conditions may be described in terms of the Liapunov exponent (Schuster, pp. 18-22, 34). Another project will be the experimental work with the R-L-diode circuit, with stipulated procedures chosen to reproduce interesting patterns that have previously been identified. Complementary to the experimental work will be computer simulation of the diode circuit, again with stipulated procedures. Finally, the students will be asked to utilize the experience gained in the structured projects to conduct some investigations of their choice, both with the experimental apparatus and by computer simulation.

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