Finding Qualitative Behavior Modes: The Use of Interval Analysis to Perform Sensitivity, Stability, and Error Analysis on Dynamic Models

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ABSTRACT

A method is described and illustrated for explicit incorporation of and computation with ranges of initial conditions, functions, and parameter values in dynamic models using interval analysis. This approach is neither a statistical nor fuzzy set analysis but instead utilizes interval arithmetic which is particularly well suited for computerization. When a dynamic model is couched in interval analytic terms, ranges of all possible solutions are generated allowing not only an analysis of ranges of behavior modes but for sensitivity and stability analysis to be performed as a natural part of the model. Moreover, uncertainties such as specification, numerical method (e.g., numerical integration), and roundoff errors can also be analyzed in conjunction with or separate from the interval dynamic model.

INTRODUCTION

In analyzing system dynamic models, time is spent in searching for the conditions, if any, which generate different qualitative behavior modes. There are a number of reasons for being interested in the possibility of the model generating two or more behavior modes. In building confidence in models--see, e.g., Forrester and Senge (1980)--one would like to reject the models if changes in parameters yield anomalous behavior that appears to be unjustifiable. Secondly, the real problem that is being modeled may be multifacited so if the system itself is known to display more than one behavior mode, the system dynamicist must show that the model can capture these known dynamics. In cases where the system is very complex and a priori knowledge about the system relatively unknown, a good model might generate surprise behavior patterns, which actually may be contained in the data, but was not discovered until the model pointed out its existence and perhaps its importance.

Another important reason for seeking methods for aiding in discovering different behavior modes is to bolster the modeler's argument against stressing the quantitative aspects of the problem, instead of paying more attention to the relationship between dynamic structure and the qualitative behavior of the system. In particular, if a modeler can show that under all reasonable ranges of parameter values, the model behaves in the same manner, and thus is insensitive to parameter changes, then the client, editor, and/or grant reviewer may feel less inclined to insist on elaborate aggregated statistical analysis of the past history of the system as the sole criterion of the model's validity.

Moreover, in recent years, there is a growing trend in the system dynamic literature to explore ways of performing sensitivity analysis on continuous nonlinear dynamic models (Tank-Nielsen 1980; Forrester 1983; Ford, Amlin, and Backus 1983; Graham and Pugh 1983). Obviously, methods for searching for different qualitative behavior modes would fall under the general heading of sensitivity analysis. Forrester (1983) and Graham and Pugh (1983) take an eigenvalue approach of linearizing the model around important operating points to assess possible behavior modes and to attempt to understand the role of each major loop in the model. On the other hand, Ford, Amlin, and Backus (1983) take a more global statistical approach to parameter sensitivity, which is somewhat similar to our method which uses interval analysis. In any event, all approaches to sensitivity analysis have to contend with several problems:

(1) In large complex models, the potential number of combinations of initial states, parameter values, and shapes of table functions is enormous. In fact as will be seen, if in a linear model there are n (constant) parameters whose values are "fuzzy" but contained in a bounded interval, there are 2^n combinations. For continuous functions or nonlinear models whose "fuzzy" domains and ranges are contained in bounded sets, there are, of course, uncountably infinite combinations.

(2) Sensitivity analysis is frequently performed in situations where one must contend with noisy input functions, roundoff and numerical (e.g., integration) error, as well as specification error.

(3) Attempting the classical strategy of keeping everything constant except one parameter or initial value may be somewhat misleading due to the interaction among parameters. Some qualitative behaviors may only show up under certain combination of parameter conditions.

We are just learning to apply interval analysis to overcome these problems. The key idea is to locate those aspects of the model which are the most uncertain or fuzzy and to represent those uncertainties in terms of bands or intervals of values. For example, in specifying table functions, usually there are a few points where, from the logic of the situation, there is no uncertainty at all. The origin or the point (1,1) frequently are set up in multipliers to cut off a rate equation when inventories are zero or to insure that the multiplier has no influence under normal baseline conditions. On the other hand, parts of table functions may be extremely fuzzy. Interval analysis allows us to put bands around the more uncertain parts of table functions before the system is simulated, and therefore brings in a form of sensitivity analysis at an early stage of the model building process.

One of the potential strengths of interval analysis is its ability to handle any combination of bands of parameter values and to generate an envelope of output for each requested variable. The information per run about potential behavior modes is maximal, and, as we will demonstrate in a simple example, it would take many more runs with the classical approach to get similar information. In addition, we will also show how interval analysis can work with combination of parameter changes, and how those results could differ from changing one parameter at a time, even when using the extreme values of the parameters in each run.

INTERVAL ANALYSIS--INTRODUCTION

Interval arithmetic is an algebraic and topological structure (Moore 1979 or Alefeld and Hertzberger 1983) which uses as its basic element of analysis closed and bounded real valued intervals $A = [a_1,a_2] = \{x: a_1 \leq x \leq a_2$ where $a_1,a_2,x \in$ real numbers$\}$. The interval A is regarded as a number in much the same way as $x = a+bi$ is regarded as a number. Interval numbers however not only possess an algebraic structure (different from real and complex numbers since the distributive law does not hold but a sub-distributive one) but, since an interval number is also a set, possesses a topological structure as well. We will be most interested in exploiting the algebraic structure as a tool in systems dynamic (SD). Since real numbers are a subset of interval numbers; that is, $A = a = [a,a]$ is a real number (an interval of zero width); the methods developed herein are an extension of the "usual" SD approaches.

The following notation will be used. When the context is clear, the capitalized letters A, B, C, X, Y, and Z denote interval variables. Otherwise, a superscript "I" is used; e.g., $a^I = [a_1,a_2]$. The capitalized letters F, G, and H will denote interval valued function counterparts to the real valued functions f, g, and h, respectively, and defined in what follows. When the context is not clear, an interval valued function is denoted as $f^I(x^I)$.

The four categories of errors to be analyzed are: (i) model specification (or simply specification), (ii) data or measurement, (iii) numerical method or discretization, and (iv) truncation. Truncation error is of two types: (iva) numerical (finite numerical representation of an infinite numerical process), and (ivb) roundoff or finite state machine errors. The methods of interval analysis, which include interval arithmetic, will be used to explicitly incorporate errors (ii), (iii), and (iv) in dynamic models. However, when quantitative (or qualitative) bounds are known for (i), it too can be analyzed. It should be clear that quantitative (or qualitative) error bounds yield quantitative (or qualitative) solutions where by quantitative bounds is meant that bounds are known precisely.

INTERVAL ANALYSIS--A SHORT REVIEW

Though interval analysis has its roots in Archimedes' method to derive an approximation for π, its formal study in numerical analysis began in 1962 when R. E. Moore published his Ph.D. thesis. An extensive literature now exists with over 757 citations in Moore (1979) alone. The interested reader is directed to the more recent Alefeld and Herzberger (1983) and Moore (1979) expositions. An extremely abbreviated synopsis is presented here.

Interval arithmetic is defined as follows. Let $A = [a_1,a_2]$ and $B = [b_1,b_2]$.

$$A+B = [a_1+b_1,a_2+b_2] \quad (1)$$

$$A-B = [a_1-b_2,a_2-b_1] \quad (2)$$

$$AB = [\min\{c\},\max\{c\}] , \quad (3)$$

where $c = \{a_1b_1,a_1b_2,a_2b_1,a_2b_2\}$. If $0 \notin B$, then

$$A/B = [\min\{c\},\max\{c\}] \ , \qquad (4)$$
where $c = \{a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2\}$.

Assuming that the algebraic operations can be carried out exactly, each operation (1)-(4) above has the property that for any $x \in A$, $y \in B$, $x \circ y \in A \circ B$, where $\circ$ denotes one of the four algebraic operations. On the other hand, for any $z \in A \circ B$, there exists $x \in A$ and $y \in B$ such that $z = x \circ y$. However, this exact interval arithmetic (EIA) will not satisfy the above remarks when implemented on a computer. The presence of roundoff error in computer floating point representation of real numbers and in computer arithmetic cannot be guaranteed to produce the exact endpoints. By analyzing the accuracy of the floating point representation and arithmetic operations on a given machine, it is possible to "round" the machine-computed endpoints and floating point numbers to insure containment. We no longer have equality as in (1)-(4). The implementation of interval arithmetic in the presence of roundoff errors of particular machines is called rounded interval arithmetic (RIA) and FORTRAN-based RIA exist for CDC, DEC, Honeywell, IBM, and UNIVAC computers (Moore 1979, pp. 14-17) as well as general ALGOL 60 RIA's (Alefeld and Herzberger 1983, pp. 288-295).

The essential condition of RIA is that $A \circ B \subset A \tilde{\circ} B$ where $\circ$ is (1)-(4) and $\tilde{\circ}$ indicates the use of RIA. It follows from monotonicity that, if F is a rational expression in the interval variables $X_1, X_2, \ldots, X_n$, then $F(X_1, X_2, \ldots, X_n) \subset \tilde{F}(X_1, X_2, \ldots, X_n)$ where $\tilde{F}$ indicates the use of RIA in the evaluation of F. Suppose that f is a real valued function defined on the interval A. Consider next the problem of finding the exact values $f(x)$, $x \in A$ on a computer. First, x might not be representable exactly and would itself have to be replaced by an interval containing x. Second, f might (usually does) involve arithmetic operations leading to roundoff errors. Third, f might contain irrational expressions and even be transcendental leading to approximation procedures when computed.

The concept of interval extension (or range) of functions is introduced to deal with these problems. The interval valued function F of n variables $X_1, X_2, \ldots, X_n$ is an interval extension of the function f if $F(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$. To simplify the exposition, we treat the one variable case, where the extension to the n variable case is easily accomplished. The range of a real valued function $f(x)$, $x \in X$, is denoted $RF = \{f(x): x \in X\}$. It is proved in Moore (1979, p. 22) that if F is a rational interval function and an interval extension of f, then $Rf \subset F$. Therefore, at least formally, there is a means of dealing with the "usual" SD models possessing rational functions and table functions. In general under some conditions on f, f can be approximated by a rational function where the bounds of approximation are well known. These bounds are added (and subtracted) to form an interval extension formulation.

However, serious problems exist in obtaining a "suitable" interval extension function F for f in the sense that F(X) is as close as possible to RF; i.e., such that $\mu[Rf-F(X)] \leq \varepsilon$ where μ is the measure (absolute value) on the real numbers, ε arbitrarily small. The following example from Alefeld and Herzberger (1983, p. 25) illustrates the problem.

Let $f(x) = x(1-x)$ where we have $F_1(X) = X-X^2$, $F_2(X) = X(1-X)$, $F_3(X) = \frac{1}{4} - (X-\frac{1}{2})(X-\frac{1}{2})$, and $F_4(X) = \frac{1}{4} - (X-\frac{1}{2})^2$ as interval extensions of $f(x)$, $x \varepsilon X = [0,1]$. $Rf = [0,\frac{1}{4}]$, and applying (1)-(3), we obtain $F_1(X) = [-1,1]$, $F_2(X) = [0,1]$, $F_3(X) = [0,\frac{1}{2}]$, and $F_4(X) = [0,\frac{1}{4}]$. As it turns out, to obtain a "minimal" interval extension, centered forms such as $F_4(X)$ above are usually used (Ratschek and Rokne 1984, pp. 30-92, 103-110). It is assumed in what follows that by interval extension F, the minimal interval extension is meant.

SENSITIVITY AND STABILITY ANALYSIS

Sensitivity Analysis: Once the dynamic model is couched in interval analytic terms and arithmetically performed using EIA or RIA, sensitivity analysis can be done very simply. This is because variations in the parameters, initial conditions, and even the error bounds themselves can be viewed as intervals. If all interval variations are zero width intervals, "traditional" sensitivity analysis is obtained. However, when one or more are intervals of width greater than zero, the full range of all possible solutions results. Therefore, interval dynamic models are an extension of "traditional" sensitivity analysis. Actually, in view of the incorporation of truncation and numerical errors, the computed solution is usually (always) a larger range of values containing the actual range of values of the solution.

Stability Analysis: There are two ways which this analysis can proceed, empirically or directly. Once an equilibrium point has been found, the empirical method takes variations in the parameters using various interval values of the parameters and executes the interval version of the dynamic model to see how the final solution is changed. Thus, the empirical method is a sensitivity analysis about the equilibrium point and can be carried out using the techniques already described.

The direct method requires an interval analytic eigenvalue solver. For example, the power method could be used in the following way. Implement the power method in the usual fashion with real numbers and usual arithmetic. Once the algorithm reaches a predetermined tolerance, perform the power method one more time using RIA and theoretical error bounds where the first inputted value in the interval power method is an interval of zero width equal to the last iterate of the real number power method. Then perform one step of the interval power method. If the QR algorithm is used, an interval version would have to be used. Lastly, if the dimension of the matrix A is not prohibitively large, then the zeros of

$$P(\lambda^I) = \det(A^I - \lambda^I I^I) \qquad (5)$$

can be solved by techniques found in Alefeld and Herzberger (1983, pp. 101-112).

TWO EXAMPLE IMPLEMENTATIONS

Salient features of the interval analytic approach to SD are demonstrated by the following two simple examples. EIA is used here merely to simplify the analysis.

A Simple, Linear One-Level Discrete Dynamic Model: To illustrate the application of interval analysis, consider

$$x(k+1) = a*x(k) + b\ , \qquad (6)$$
$$x(0) = 1.0\ ,\quad a = -1.0\ ,\quad b = 4.0\ ;\quad k = 0,1,\ldots,5\ ,$$

whose properties are known (Goldberg, 1958).

This is an extremely simple difference equation which is not often seen or used in system dynamics. First it should be noted that the model with those specific values of a and b will generate oscillatory behavior, even though it is of first order. This points out the qualitative differences between the dynamics of differential and difference equations. Oscillatory behavior does not emerge in a first-order differential equation system, but can arise in a system described by a difference equation, such as the one above. A first-order difference equation might be useful in describing how people manage systems in which the only information comes at a sampled interval. For example, the second author has worked with doctors who get information about their diabetic patients on a daily basis. A poor doctor might only pay attention to one state variable, such as amount of sugar in the blood. If the sampling rate is too slow, the doctor can cause tremendous oscillations in the patient by prescribing more insulin to counter the rising blood sugar level.

The output of this oscillatory model would be 3.0, 1.0, 3.0, 1.0, etc. How would one apply interval analysis to this simple model? The first step is to define the intervals around each of the parameters and the initial value of x.

Suppose that $a \in A = [a_1, a_2]$, $b \in B = [b_1, b_2]$, and $x(0) \in X(0) = [x_1(0), x_2(0)]$. Then, $X(k+1) = A*X(k) + B$, where $X(k) = [x_1(k), x_2(k)]$ and $*$ and $+$ are performed using (3) and (1). Given variations so that $a_2 \leq 0$ and $b_1 \geq 0$, the interval model becomes

$$x_1(K+1) = \begin{cases} a_1*x_2(k) + b_1 & \text{if } x_2(k) \geq 0 \\ a_2*x_2(k) + b_1 & \text{if } x_2(k) < 0 \end{cases}$$
$$x_2(k+1) = \begin{cases} a_2*x_1(k) + b_2 & \text{if } x_1(k) \geq 0 \\ a_1*x_1(k) + b_2 & \text{if } x_1(k) < 0 \end{cases} \qquad (7)$$

Let $A = [-1.1, -0.9]$, $B = [3.9, 4.1]$, and $X(0) = [0.9, 1.1]$; then (7) yields Table 1, where the lower bound $x_1(k)$, upper bound $x_2(k)$, center $= \frac{1}{2}(x_2(k) - x_1(k))$, and width $= (x_2(k) - x_1(k))$ of the generated interval $[x_1(k), x_2(k)]$ are listed. The interval $[x_1(k), x_2(k)]$ in Table 1 is the envelope of all possible trajectories generated by the simultaneous variations in the parameters a, b, and initial value x(0) given by A, B, and X(0). To attempt to duplicate the information gained in the <u>one</u> implementation of the interval model (7), the classical approach might be performed six times, taking the extreme value at each end of the three variations A, B, and X(0) and keeping all other values constant. This is done and listed in Table 2.

k	1	2	3	4	5	6	...	24	25
$x_1(k)$	2.69	0.281	2.0531	-0.3318	1.42257	-1.01149	...	-18.8693	-9.76447
$x_2(k)$	3.29	1.679	3.8471	2.25221	4.46499	2.81969	...	12.4222	24.852
Center	2.99	0.98	2.9501	0.96020	2.94378	0.9410	...	-3.22352	7.54587
Width	0.6	1.398	1.794	2.58402	3.04242	3.83118	...	31.2915	34.6207

TABLE 1--Envelope of Trajectories

The six classical sensitivity analyses for (6) would be to consider the following substitutions one at a time holding other values at their original values: a = -1.1, a = -0.9, b = 3.9, b = 4.1, x(0) = 0.9, and x(0) = 1.1, which would be A = [-1.1,-1.1], B = [4.0,4.0], X(0) = [1.0,1.0]; A = [-0.9,-0.9], and so on when (7) is used. The results are (min and max values underlined):

k	1	2	3	4	5	6	...	24	25
a = -1.1; x(k)	2.9	0.81	3.109	0.5801	3.36189	0.30192	...	-7.0069	11.7076
a = -0.9; x(k)	3.1	1.21	2.911	1.3801	2.75791	1.51788	...	2.0171	2.18461
b = 3.9; x(k)	2.9	1.0	2.9	1.0	2.9	1.0	...	1.0	2.9
b = 4.1; x(k)	3.1	1.0	3.1	1.0	3.1	1.0	...	1.0	3.1
x(0) = 0.9; x(k)	3.9	0.9	3.1	0.9	3.1	0.9	...	0.9	3.1
x(0) = 1.1; x(k)	2.9	1.1	2.9	1.1	2.9	1.1	...	1.1	2.9
Center	3.0	1.01	3.0045	0.9801	3.0599	0.9099	...	-2.4949	6.9461
Width	0.2	0.4	0.209	0.8	0.604	1.216	...	9.0240	9.523

TABLE 2--Six Classical Sensitivity Analyses

If simultaneous variations of (6) were executed with the left and right bounds of A, B, and X(0); i.e., using a = -1.1, b = 3.9, and x = 0.9, and then using a = -0.9, b = 4.1, and x(0) = 1.1; we obtain the following:

k	1	2	3	4	5	6	...	24	25
x(k)	2.91	0.699	3.1311	0.4558	3.3986	0.1615	...	-7.5705	12.2275
x(k)	3.11	1.301	2.9291	1.4638	2.7826	1.5957	...	2.0735	2.2338
Center	3.01	1.0	3.0301	0.9598	3.0908	0.8786	...	-2.7485	7.2307
Width	0.2	0.602	0.202	1.0080	0.6163	1.4342	...	9.644	9.9937

TABLE 3--Simultaneous Variations

Note that model (6) implementations were unable to capture the full effect of the interaction of the errors as can be seen by comparing Table 1 to Tables 2 and

3. To capture the complete interaction of three pairs of variations $[a_1,a_2]$, $[b_1,b_2]$, and $[x_1(0),x_2(0)]$ of the constant parameters a, b, and x(0), eight runs of (6) must be made and in general a linear model with n interval variations of (constant) parameters requires at least 2^n runs to guarantee the capture of the full effect of interactions of the n variations. It requires one run of the interval model to obtain the same information. When the model has a continuous function which changes monotonicity over the variations and subsequent generated values of the state variable, more than 2^n runs (in fact an uncountably infinite number in many cases) would be required to guarantee one had obtained all possible interactions of the variations, but requires one run of the interval model.

A Predator-Prey Model--The Kaibab Plateau Model (Goodman 1980, pp. 377-388): The second example presented is closer to home. This model has been used frequently in the system dynamics literature for teaching purposes and a DYNAMO version can be found in Goodman (1980). There are three levels in this model: deer, predators, and food. The inputs are total area (AREA) = 800,000 acres, average food per deer (AFPD) = 1.0 units, and the removal rate of predator (RF). The time period starts in 1900 with the deer population (DP) = 4000, predator population (PP) = 8000, food capacity (FCAP) = 350,000, and available food for deer (F) = 350,000 units. In 1905 a bounty on predators is given so that RF = 0.2 from 1905 on. The time increment (DT) = 0.1 is used.

The equations of the model are:

	Food per deer	FPD = F/DP	(8)
	Food ratio	FR = FPD/AFPD	(9)
(1)	DEER SECTOR		
	Deer growth rate factor	DGRF = DGRFT(FR) Table Function	(10)
	Deer net growth rate	DNGR = DP*DGRF	(11)
	Deer density	DD = DP/AREA	(12)
	Deer kill ratio	DKR = DKRT(DD) Table Function	(13)
	Deer predator rate	DPR = PP*DKR	(14)
(2)	PREDATOR SECTOR		
	Predator growth rate factor	PGRF = PGRFT(DKR) Table Function	(15)
	Predator net growth rate	PNGR = PP*PGRF	(16)
	Predator bounty removal	PBR = PP*RF	(17)

where RF = 0.0 for 1900 until 1905 and RF = 0.2 from 1905 on.

(3)	FOOD SECTOR		
	Food capacity fraction	FTFCAP = F/FCAP	(18)
	Food regeneration rate (years)	FRT = FRTT(FTFCAP) Table Function	(19)
	Growth rate (units)	GR = (FCAP-F)/FRT	(20)
	Food consumption rate per deer	FCPD = FCPDT(FR) Table Function	(21)
	Food consumption (units)	FC = DP*FCPD	(22)
(4)	INCREMENT		
	Deer population	DP: = DP + DT(DNGR-DPR)	(23)
	Predator population	PP: = PP + DT(PNGR-PBR)	(24)
	Food available	F: = F + DT(GR-FC)	(25)

The interval analytic model at first glance would appear to be a straightforward substitution of each variable with interval variables, each algebraic operation with RIA (or as in our example EIA), and each table function with an interval table function. However, if this were done, intervals whose widths are too large ensue. This is most clearly seen in (17) and (24). Let the interval containing PBR at period k be denoted [PBR1(k),PBR2(k)] and likewise for the other variables. Since $P \geq 0$ and $RF \geq 0$, using interval multiply (3), (17) becomes:

$$PBR1(k) = PP1(k)*RF1(k) \tag{26}$$

$$PBR2(k) = PP2(k)*RF2(k)\ . \tag{27}$$

Using interval add (1) and subtract (2), (24) in a straightforward substitution becomes:

$$PP1(k+1) = PP1(k) + DT(PNGR1(k) - PP2(k)*RF2(k)) \tag{28}$$

$$PP2(k+1) = PP2(k) + DT(PNGR2(k) - PP1(k)*RF1(k))\ . \tag{29}$$

However, the interval arithmetic is removing a predator bounty kill PP2(k)*RF2(k) associated with the largest value of the predator population PP2(k) from the smallest value of the predator population PP1(k). Clearly, PP1(k) does not experience that removal PP2(k)*RF2(k). It does however experience a maximal removal of PP1(k)*RF2(k). For PP2(k+1), the smallest removal of PP2(k) by bounty hunters is required coupled with the largest growth of the predator population. Thus, the correct interval for (24) is:

$$PP1(k+1) = PP1(k) + DT(PNGR1(k) - PP1(k)*RF2(k)) \tag{30}$$

$$PP2(k+1) = PP2(k) + DT(PNGR2(k) - PP2(k)*RF1(k))\ . \tag{31}$$

There are two approaches to converting a system dynamic model into an interval model when it is represented as in this example, by a list of interconnected equations. The first way would be to analyze the set of equations carefully to determine how the endpoints of the interval variables are to be formed as was done for (30) and (31) above. The second approach would be to state the model in its vector function representation which for the discrete-time model is:

$$\vec{x}(k+1) = \vec{f}(\vec{x}(k);k)\ . \tag{32}$$

Then transform (32) into its interval representation recalling that interval function evaluation must be performed. That is, (32) becomes:

$$\vec{X}(k+1) = \vec{F}(\vec{X}(k);k)\ . \tag{33}$$

The transformation of (8)-(25) into an interval model is given below where the minima and maxima are taken with respect to the (constrained) state variables $DP(k) \in [DP1(k),DP2(k)]$, $PP(k) \in [PP1(k),PP2(k)]$, and $F(k) \in [F1(k),F2(k)]$.

$$\begin{aligned} DP1(k+1) &= \min\{DP(k) + DT[DNGR(k) - DPR(k)]\} \\ &= \min\{DP(k) + DT[DP(k)*DGRFT(FR(k)) - PP(k)*DKRT(DD(k))]\}\ , \end{aligned} \tag{34}$$

where $FR(k) = F(k)*AFPD/DP(k)$ and $DD(k) = DP(k)/AREA$. When average food per deer AFPD and/or area are themselves positive width intervals, then the minimum in (34) must also include these two (constrained) variables. Moreover, if or when the left end of the deer population DP1(k) goes to zero, a suitable interpretation of FR(k) must be made or the model halted. The right end of the deer population thus becomes:

$$DP2(k+1) = \max\{DP(k) + DT[DP(k)*DGRFT(FR(k)) - PP(k)*DKRT(DD(k))]\}\ . \tag{35}$$

Likewise,

$$PP1(k+1) = \min\{PP(k) + DT[PP(k)*PGRFT(DKR(k)) - PP(k)*RF(k)]\} \tag{36}$$

$$PP2(k+1) = \max\{PP(k) + DT[PP(k)*PGRFT(DKR(k)) - PP(k)*RF(k)]\}\ . \tag{37}$$

When $RF(k) \in [RF1(k),RF2(k)]$ is an interval of positive width, then the min/max above must include it in addition to those of the three state variables. Here $DKR(k) = DKRT(DD(k))$ and PGRFT(DKR(k)) is a function of a function; i.e., a composite function.

$$F1(k+1) = \min\{F(k) + DT[GR(k) - DP(k)*FCPDT(FR(k))]\} \tag{38}$$

$$F2(k+1) = \max\{F(k) + DT[GR(k) - DP(k)*FCPDT(FR(k))]\}\ , \tag{39}$$

where GR(k) = (FCAP - F(k))/FRTT(F(k)/FCAP). As before, when FCAP belongs to a positive width interval, then the min/max must incorporate these variations. In (34)-(39) DT, the time increment, is considered as a zero width interval.

What happens then when we transform (8)-(25) "blindly" into an interval model as illustrated by (27)-(29)? When a straight substitution is made, it would be equivalent to moving the min/max of (34)-(39) inside the parentheses; that is, we are obtaining an interval, while guaranteeing to contain all possible trajectories due to the variations, which is nevertheless far too wide, rendering potentially "meaningless" results. The interval model (34)-(39) takes min/max in tandem. For example, consider what occurred in (26) and (28) and compare this to (36). The minimum in (36) is taken as PP(k) varies over [PP1(k),PP2(k)] (and the other state variables of course) so that the worse case interval PNGR1(k) - PP2(k)*RF2(k) = PP1(k)*PGRFT(DKR(k)) - PP2(k)*RF2(k) of (28) would not occur for (36) since PP(k) will be the same value throughout (36). Clearly, (34)-(39) require a range function analysis or an analytic min/max solution. However, before computing, the following simplification can/should be made:

DP(k+1) = min{DP(k) + DT[DP(k)*DGRFT(F1(k)*AFPD1/DP(k))
- PP2(k)*DKRT(DP(k)/AREA2)]} , (34')

where DP(k) ε [DP1(k),DP2(k)].

DP2(k+1) = max{DP(k) + DT[DP(k)*DGRFT(F2(k)*AFPD2/DP(k))
- PP1(k)*DKRT(DP(k)/AREA1)]} , (35')

where DP(k) ε [DP1(k),DP2(k)].

PP1(k+1) = min{PP(k) + DT[PP(k)*PGRFT(DKR1(k)) - PP(k)*RF2(k)]
= min{PP(k)[1 + DT(PGRFT(DKR1(k)) - RF2(k))]}
= PP1(k)[1 + DT(PGRFT(DKR1(k)) - RF2(k)]} , (36')

where PP(k) ε [PP1(k),PP2(k)].

PP2(k+1) = PP2(k)[1 + DT(PGRFT(DKR2(k)) - RF1(k)]} . (37')

Equations (38) and (39) do not simplify due to the nonlinearity of GR(k) and DK(k)*FCPDT(FR(k)).

CONCLUSIONS

An analysis of Tables 1, 2, and 3 indicates that interval analysis is able to capture the full effect of parameter uncertainty in one pass which would require the full range of parameter permutations for sensitivity. Secondly, once a set of equations is put into an interval analytic setting, which in view of (26)-(29) is not always a one-to-one correspondence between algebraic operations, the full range of sensitivity analysis can be performed either in the usual fashion or in a way which analyzes effects due to multiple variations. One is assured of obtaining the full range of permuted variations in a single analysis.

Secondly, it is clear from the second example, (8)-(37'), that care must be taken in transforming a model into an interval analytic setting. Whether the effort involved in doing this is worth the gain accrued in being able to model explicitly parameter, numerical, and roundoff errors will, of course, depend on the purpose for which the model serves. However, it has been demonstrated that the methods of interval analysis can be used to perform just such a task.

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