# IDENTIFYING AND DISPLAYING IMPORTANT FEEDBACK PATHS

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Abstract. Fundamental to the practice of system dynamics is the identification of feedback. The theory of linear model analysis and model simplification provide tools for doing this in the setting of linear state space models. The application of these tools in the field of system dynamics has been very limited primarily because the tools are inaccessible and difficult to use. Many of the difficulties can be overcome by linking the analysis more closely with the original nonlinear model. We do this first by using time plots of model variables to describe behavior and second by deriving a nonlinear feedback model that can be used to exhibit the important feedback structure. The theory for doing this is heuristic, but allows the techniques to be automatically applied with interaction only in the domain of the original nonlinear model and its simulation.

## INTRODUCTION

The identification and understanding of feedback is fundamental to the practice of system dynamics. Though there is a good deal of theory dealing with this problem, the theory has had only very limited applications. The primary reason for this is the inaccessible nature of the work that has been done. In this paper we will develop some rules of thumb that can be used to perform the process of identifying feedback. This combined with an easily understood presentation of the problem and results allows widespread access to some very useful tools.

The process of system dynamics involves the analysis of models. At the core of this analysis is the determination of what feedback paths in a model are responsible for the behavior of interest. The tools we discuss in this paper are intended to aid in this analysis. These tools require that the definition of behavior be restricted and carefully specified.

In most cases behavior of interest is a composite of less interesting but more basic behavior. For example, the behavior of interest might be a lack of innovation, which results from instability in the ratio of experienced to inexperienced staff. The instability in the staff is of less interest, but can more easily be cast into framework of behavior we will discuss in this paper. The process of casting something of interest into something that can be analyzed is often tortuous and not always successful. To aid in this we will derive an approximate representation of things that can be understood, in the hope that connections can be made to those that need be understood. Nonetheless, successful application of tools being described here still requires a solid understanding of the model under consideration.

Richardson (1986) has outlined a number of different dimensions along which the analysis of feedback structure occurs. The three basic frameworks identified are: the traditional methods of repeated simulation with experimentation, consideration of how the model output is determined by different time patterns of inputs, and analysis of the eigenvalues of a model. We deal only with the last of these in this paper, but tie the work that is done to the more traditional methods involving time plots. Although the theory is based on the assumption of linearity, we will discuss applications to nonlinear models.

## BEHAVIOR

Behavior is a very general concept which must be made specific, and thus quantifiable, in order to use it as a basis of analysis. The concept of behavior we use has its foundation in the theory of linear analysis of dynamic models. In order to explain this concept we will first show how we go from a nonlinear feedback model to the linear form that we need. Then we will discuss what the behavior we are considering means in the context of the linear model and extend this back to the nonlinear model.

A feedback model consists of levels, which accumulate over time, and rates and auxiliaries that depend on the levels, and determine how the levels change. The linear model we base our analysis on has only levels, and the rate of change in a level is specified as a linear combination of all the levels. We use a linearization that is an approximation of the nonlinear model of interest. The method used to arrive at this approximation is quite intuitive. In a linear model changing state 1 by δ will cause a proportionate change in the rate that integrates into state two. That proportion is used as the coefficient for state 1 in the equation determining the rate at which state 2 changes.

For example, suppose that the equation for LEV1 is given by

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L    LEV1.K=LEV1.J+DT*R1.JK                                    (1)
R    R1.KL=.3*LEV1.K*LEV2.K
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with LEV1 having a value of 10 and LEV2 a value of 100. If we change LEV1 by δ we see that R1 changes by .3*δ*100, this gives a proportionate change of 30. Similarly if we change LEV2 by δ then R1 changes by .3*10*$\delta$ giving a proportionate change of 3. Thus the linear equation for LEV1 would be

L LEV1.k=LEV1.J+DT*(30*LEV1.J+3*LEV2.K) . (2)

The actual linear equation clearly depends on the values of levels and will be different at different times during the simulation.

When the procedure we have discussed is used to compute a model linearization the result is in the form

$$\dot{\underline{x}} = \underline{A}\,\underline{x} \tag{3}$$

with $\underline{A}$ the matrix with coefficients determined by the technique discussed above. Exogenous variables have been left out of equation 3 for simplicity. The matrix $\underline{A}$ will be referred to as the dynamics matrix. In this equation $\underline{x}$ represents the vector of levels in the model, $\underline{x}$ will also be referred to as the state vector.

Given a linear dynamic model in the form of equation 3 it is difficult to determine in advance what behavior may result. To aid in this, the dynamic system of equation 3 can be transformed to a series of scalar differential equations. These scalar equations have easily determined dynamics and can be used to determine what the dynamics of the original system are. The transformation of the model given in equation 3 involves both a transformation of the $\underline{A}$ matrix and of the state vector $\underline{x}$.

The matrix $\underline{A}$ can be transformed into a diagonal matrix having the eigenvalues of $\underline{A}$ along the diagonal. These eigenvalues represent the behavior generated by the (linear) model in the following sense. If $\lambda$ is an eigenvalue of $\underline{A}$ then, in the absence of any external influence, there is a scalar linear combination of the states $\underline{x}$ (that we will call $\xi$) having the property

$$\dot{\xi} = \lambda\,\xi \tag{4}$$

That is, $\xi$ changes over time in a way that is proportional to itself. The eigenvalue $\lambda$ determines the behavior of the variable $\xi$ over time. If $\lambda$ is positive then there will be exponential growth, if negative exponential adjustment and if complex there will be oscillations. For a model with N states there will be N equations in the form of equation 4.

Because the levels in the model can be written as a linear combination of the different $\xi$'s the dynamics of the system given in equation 3 can be described by the different eigenvalues. How the dynamics defined in this manner manifest themselves in the original states depend on what linear combination of the $\xi$'s yields the state variable. This linear combination is given by the appropriate entries of the right eigenvectors of the matrix $\underline{A}$ (Chen 1970 and Porter and Crossley 1972 give a detailed description of this decomposition).

The eigenvalues and eigenvectors of a matrix provide a useful method for breaking down the dynamics of a complex feedback system. The basic building block of this analysis is the linear transformation of the states to the eigenstates (the $\xi$'s), and back again. This transformation can be made at any time, so that the time path of the eigenstates can be used to determine the time path of the model states. We use the idea of the eigenstates in order to break down the time paths of the states into different components. This approach allows us to easily present the contribution of a behavior mode to a states overall behavior.

## Showing a Mode

The discussion of modes as being represented by the eigenvalues of the $\underline{A}$ matrix does not lend itself to easy inspection of what the eigenvalues really mean. Eigenvalues do imply certain patterns of behavior (damping times and periodicities), but modes can combine in the model in a variety of ways. We base our presentation of behavior modes on the time paths of the model variables. The presentation of behavior modes we describe has a strict theoretical basis only in the case of the linear system discussed above, but can also be applied to nonlinear systems.

We apply a decomposition to the time paths of all the variables of the original nonlinear model. This decomposition is based on a linearization of the model as discussed above. The values of the states from a nonlinear model simulation are used to generate values for the eigenstates (the $\xi$'s) by pre multiplying by the matrix of left eigenvectors. That is,

$$\xi_k = (\underline{l}_k)^T \underline{x} \qquad (5)$$

where $\underline{l}$ is the left eigenvector written as a column vector. The contribution of a mode to the current value of a state is given by multiplying the eigenstate by the right eigenvalue as in

$$(\underline{r}_k)_j \, \xi_k \qquad (6)$$

for state j in the k'th mode. The above calculation assumes that the right and left eigenvectors are normalized to have an inner product of one. When the eigenvalue is complex twice the real part of the associated calculation is used. Because the transformations involved are invertible adding up all of the contributions thus calculated will yield the original behavior.[1]

The above manipulations allow us to break down the behavior of a state into components attributable to different modes. For the linear model of equation 3 in the absence of any exogenous inputs those components would be determined by equation 4. These components would be exponential growth, exponential adjustment, damped oscillation or expanding oscillation. When exogenous variables are impacting the model (as is normally the case) the behavior attributable to modes will not be quite so straightforward. The exogenous variables might, for example, introduce a changing goal for an exponentially adjusting mode. Thus, much of the apparent behavior due to the mode would arise because of the sensitivity of the mode to external inputs.

We can break down the behavior of variables into components due to modes, but the key requirement for an understandable presentation is that each variable have a short list of influential modes. Most variables in a model will have a large number of eigenstates influencing their behavior. This large number is reduced by only considering those that have the potential to cause a substantial portion of the variation in the variable of interest. A mode will influence behavior strongly only if the eigenstate associated with the mode is changing, and the component of the right eigenvector corresponding to the state is large. To quantify this we take the variance of the eigenstate ($\xi_k$) over the simulation and multiply by the absolute value of $(\underline{r}_k)_j$ (the entry for state j in the eigenvalue for mode k). If the result of this computation is larger than some fraction (say 30%) of the variance of the state then the mode can be considered influential enough to display.

Mechanically, the above decomposition gives first a list of modes important in determining a variables behavior, then a decomposition of the variables behavior into components attributable to the different modes. Thus if there is interest in determining why a variable displays a particular pattern of behavior one can associate that behavior with a mode or modes. This association puts us into a framework of analysis that is more exact, and thus programmable, than the standard analysis of time plots. The approach described above requires approximation and is not very

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1. Note that this is only true for the states themselves since other variables will be approximate by virtue of linearization. Model variables that are not themselves states can have the appropriate right eigenvector determined by taking a linear combination of right eigenvectors for the model states.

robust, but it bridges the gap between the traditional time plot and the less well known eigenvalue.

## LINEAR MODEL SIMPLIFICATION

The intention in this paper is to outline techniques for identifying important feedback paths. We develop these techniques based on the theory of model simplification. In the context we have been discussing a simplified model has fewer states than the original model. Because there are fewer states there are fewer potential feedback paths. What remains within the simplified model are the important states, and hence the important feedback paths.

The process of simplifying a model involves removing such important feedback paths from the framework of the large model and imbedding them in a smaller model. The small model can, thus, be used as a means of displaying the important feedback paths. The only substantive difference between a full simplification and the process we are discussing is the adjustment of the parameters of the simplified model. We shall not pursue this final stage of simplification but in other regards the language and literature we use is that of simplification.

We have outlined what we mean by a behavior mode, and now seek a structure that generates this behavior. In the language of simplification we would like a model with a smaller number of states than the original model that still generates this behavior mode. This process of simplification requires approximation, not every variable in the original model will be in the simplified model. Variables with little importance to the behavior modes of interest will be omitted. A model is simplified by keeping important variables and discarding unimportant variables. This process is inherently judgemental, and the rules we discuss have this judgment built into them.

The simplification of dynamic models has a long history stemming largely from an early desire to ease computation and allow aggregation. There are a variety of approaches to simplification depending on purpose and Genesio and Milanese (1976) and Sandell et al (1978) give extensive reviews of the literature in the area. The early work on simplifications intended to retain eigenvalues was done by Davison (1966) and Marshall (1966). In this work the emphasis was on dominant modes, modes that seemed to be the most important to determining behavior. The selection of dominant modes varies depending on purpose (Gopal and Mehta 1982, Mahmoud and Singh 1982). For the analysis done here we replace dominant modes by modes of interest, where these can be selected in the above discussed framework.

Model simplification based on selected eigenvalues was addressed by Perez Arriaga (1981) who developed rules for selecting what states to include in a simplified model. The concentration in this theory is on retaining interpretability, that is, on having a simplified model that sense can be made of. Forrester (1982) has applied these techniques in order to identify important feedback paths. A more formal analysis of the problem of model simplification is contained in Eberlein (1984). In all these works the step to actual application in a system dynamics model is long and difficult. The theory is developed for state space models in the form of equation 3 and the transformation of a system dynamics model to this form loses much of the original model meaning. We will show ways to tie the analysis back to the original nonlinear model structure, but first we review the existing analysis.

Though it is clear that the linear state space representation of a dynamic model is removed somewhat from the original model the variables of the state space representation are all contained within the original model. It is essential that the variables of the simplified model also make sense relative to those in the original model. The key requirement of a simplification intended to display structure is that the structure retained in the simplified model make sense. It is always possible to derive a simplified model containing a behavior mode, we did so in presenting the one variables eigenstate model of equation 4. Such a model does not have any useful interpretation in terms of variables of interest. The one variable in the eigenstate model is a complicated linear combination of the states of the original model.

This process of simplification normally requires approximation and this approximation can have two faces. Either the variables of the simplified model may be transformed so as to differ from those of the original model, or the model may produce different dynamics. In both of these cases displaying the feedback causing a behavior mode is an approximation in the sense that some variables contributing to a mode are likely to be ignored. The actual modes generated by a simplified model may be different because feedback having a small effect on the mode is ignored. Alternatively, if the modes are the same, the variables may have a different interpretation because they have incorporated the effects of other variables in their internal feedback. Faced with this required approximation it is necessary to identify the "best" simplification, thereby the most important feedback paths.

In order to identify the best way to simplify a model we use the concept of an exact simplification. A model is said to be exactly simplifiable if a separate block of equations with no feedback to the rest of the model can be identified and it produced the modes of interest. Rewriting equation 3 by partitioning the state vector as

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{vmatrix} \begin{vmatrix} \underline{x}_1 \\ \underline{x}_2 \end{vmatrix} , \tag{7}$$

exact simplification requires that $\underline{A}_{12} = \underline{0}$ or $\underline{A}_{21} = \underline{0}$. If the upper left hand block generates the modes of interest then the model

$$\underline{\dot{x}}_1 = \underline{A}_{11}\underline{x}_1 \tag{8}$$

can be used as the exact simplification. Most models are not exactly simplifiable, but the characteristics of an exactly simplifiable model identify the desirable features of a simplification. Importantly they identify which variables should be retained in the simplification.

The logic behind the theory of identifying important states is somewhat backward. We first consider what the properties of the simplification based on the selected states would be. Given these properties we can measure how good a simplification it is relative to an exact simplification. This will tell us how good a choice the included states were. If they are to be practicable, the actual rules for selecting states need to be applicable before the state selection is made. This will be the case for the rules we consider, though there are some issues that do not break out this easily (see Eberlein 1984 chapter 3). Our goal is to break down the states $\underline{x}$ into two components, $\underline{x}_1$ which we shall refer to as the first block variables and $\underline{x}_2$ which we shall refer to as the second block variables.

The eigenvector matrices for an exactly simplifiable model have characteristics analogous to those of the $\underline{A}$ matrix. If the block of equations generating the modes of interest is not influenced by a second block of equations ($\underline{A}_{12} = \underline{0}$) the values of the eigenstates associated with the modes of interest are independent of variables in the second block. Thus, when we decompose the behavior of the variables, the components due to the modes of interest will be independent of the behavior of variables in the second block. This means that we could give the second block variables any values and still get the modes. On the other hand, if the first block of equations does not influence the second block ($\underline{A}_{21} = \underline{0}$) none of the modes of interest will be apparent in the second block. The contribution of the modes of interest to the behavior of variables in the second block will be zero.

The above characterization of an exactly simplifiable model can be rephrased in terms of characteristics of the right and left eigenvectors. The right eigenvectors determine how much a variables displays a mode, the left eigenvectors how much a variable excites a mode. This interpretation has its basis in equations 5 and 6 which show what a variables does to an eigenstate and what an eigenstate does to a variable. Variables that are important to the generation of a mode must both influence, and be influenced by, the eigenstate; they must have, in some sense, feedback through the mode. Because of this it does not suffice to look at either the right or left eigenvectors in isolation; both must be taken into account when judging the importance of a state.

For an exactly simplifiable model the second block variables will not have feedback through the modes of interest. That is, they will have only zero components in the left eigenvectors of the modes of interest or only zero components in the right. These zero components mean, respectively, that the existence of a variable does not affect the modes or that the modes do not affect the variable. In either case it is clear that the variable is not key to the mode.

It is important to note that a variable that is not key to generating modes may still display them very clearly. The relevant issue in the generation of a mode is what happens when the variable is taken away. For example, consider a variable that does not feed back into the rest of the model. The variable may display a mode very clearly, but its removal changes nothing. Such a variable is not generating the mode in the sense we are discussing.

Given the importance of both the left and right eigenvectors it does not suffice to look at either in isolation; both must be taken into account when judging the importance of a state. The general concept in measuring the importance of a state is based on the matrix formed by summing the outer products of the right and left eigenvectors over the modes of interest. More precisely, we define the generalized participation matrix $\underline{G}$ as

$$[\underline{G}]_{ij} = \sum_k (\underline{l}_k)_i (\underline{r}_k)_j \quad , \qquad (9)$$

where i and j are indices corresponding to the states and the summation on k is over the modes of interest. For an exactly simplifiable model $\underline{G}$ has the structure

$$\underline{G} = \begin{vmatrix} I & 0 \\ * & 0 \end{vmatrix} \quad \text{or} \quad \underline{G} = \begin{vmatrix} I & * \\ 0 & 0 \end{vmatrix} \qquad (10)$$

with the * denoting some potentially nonzero matrix and I the identity matrix.

It is worth noting that in the case in which there is only one mode of interest the matrix G is the matrix of derivatives of the eigenvalue with respect to the corresponding entry in the A matrix (Porter and Crossely 1972). Also, in this case, the diagonal elements of G are what Perez Arriaga (1981) termed the participation factors. We base our selection of states on the matrix G, with three attendant measures of a states importance. First we consider the values of the diagonal elements. These will sum to the number of modes selected, and large values indicate importance. The diagonal elements have the advantage of being dimensionless and thus independent of the choice of units in the original model.

The other measures considered involve off diagonal elements and some adjustment for the units of measurement is required. We do this by multiplying each element by the corresponding element of the A matrix. This is a measure closely related to the eigenvalue elasticity and when this is done for an exactly simplifiable model only the upper left hand block of G will have nonzero elements. Any nonzero elements indicate an important feedback link involving two states. To get a summary of the important states, we take the sum of the absolute values across a row and compare this to the sum of the absolute values of all the entries in the matrix.[2] When the ratio is large it indicates the state corresponding to the row to be important. An exactly analogous calculation is made relative to columns.

The three calculations outlined above will yield the same results for an exactly simplifiable model. When the simplification is an approximation as it normally will be there may be some differences. To deal with these we take the union of all indicated variables.. The sensitivity for cutting off variables is not strongly indicated by any theory. We have used a critical value of between .25 and .5 when comparing absolute values as outlined above. The result of all of this is a list of what variables to include in the simplified model. These variables are indicated because the feedback paths among them are important for generating the modes. Thus we have identified important paths.

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2. It is straightforward to show that the row sum of elements of the weighted G matrix will equal the product of the right and left eigenvector entries of the eigenvectors corresponding to the row times the eigenvalue, all summed over all included eigenvalues. Thus, the sum of all entries will simply be the sum of the eigenvalues. The analogous result holds starting with column sums.

## DISPLAYING FEEDBACK

The display of feedback requires that the analysis move back to the domain of the original nonlinear model. Thus far the identification of feedback structure has been for the linear state space representation of the model. In this section we outline a practical approach to going from the identification of structure in state space to the identification of the structure in the model. The amount of structure retained by the techniques we discuss is likely to be somewhat excessive, but the techniques have some valuable attributes. Foremost among these is the ability to get the same model simulations by adding exogenous values for the excluded model variables.

The selection process we outlined in the previous section will cause some states to be retained and others to be discarded. An important fact about feedback paths is they all pass through states. Thus the removal of a state is essentially the same as the cutting of feedback links. Given that we have removed states, it is at the states that the breaking of feedback links begins. We replace each of the removed states by an exogenous variable, then step back to look at the results. Variables that are not used are simply removed, variables that are used, but not themselves in feedback paths can be made exogenous since they depend only on exogenous variables. Making such variables exogenous will likely leave other variables unused and, thus, allow their removal.

This process of removing variables by making other variables exogenous continues until all endogenous variables are part of the feedback structure, and all exogenous variables are used directly in defining these. An example of this process is given in the simple two state predator prey model of Figure 1. If we choose to remove the wolf population and retain the deer population we first cut the links going into the wolf population. This leaves the rodent population unused and we therefore remove this. Further, it is not the wolf population per say, but the wolf roaming that determines contacts. Thus we remove the wolf population altogether. The deer-wolf contacts depend on the deer population and cannot therefore be removed. The resulting model contains deer population, available food, deer kills and deer wolf contacts. Carrying capacity and wolf roaming enter as exogenous variables.

The model that results from the simplification just described contains the feedback essential to the modes of interest. It will not necessarily produce these modes however. To get the model to generate behavior modes adjustments would have to be made to the model parameters. We will not pursue this issue since our primary interest is just in having the essential

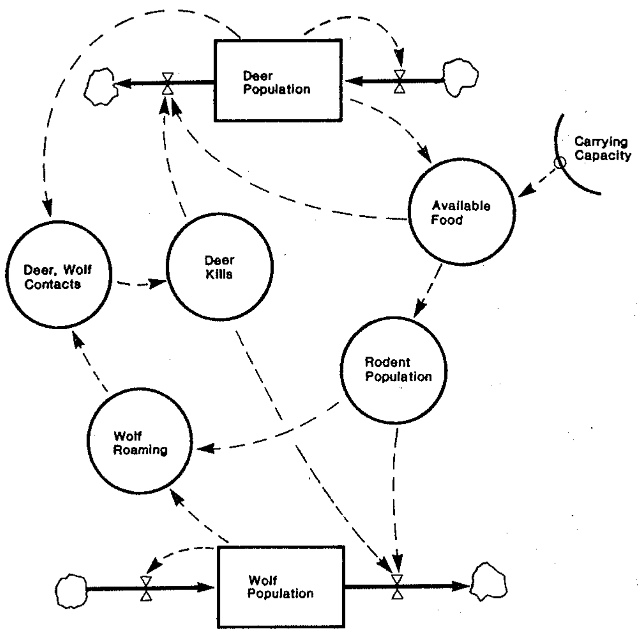


Figure 1. A Feedback in a simple predator prey model.

feedback structure. However, if the variables that we have now made exogenous are given the same values as they had in the full model simulation the simplified model will reproduce the original model behavior.

While the simplified model can reproduce the original behavior the relationship between making changes in the original and simplified model is not as clear. The goal of a simplification is to arrive at a model where changing parameters has the same effect as in the original model. To do this, however, it is necessary to alter the parameters of the simplified model. We again do not pursue this, but feel it important to recognize this restriction.

## APPLICATION TO A MODEL OF WORKER BURNOUT

In order to illustrate the techniques described we will apply them to a model of burnout as presented in Homer (1985). This model describes how the process of working at an activity can easily lead to cycles of high and low productivity and effort. The model contains only four states in the active feedback structure: energy level, hours worked per week, perceived accomplishments per week and expected accomplishments per week. Because the model is small to start, a dramatic reduction in size cannot be expected, but the model does serve to illustrate the techniques. The model listing and a complete description and behavior analysis are contained in Homer (1985).

Because the model is nonlinear the linearization yields different results at different times. These distinct linearizations can be viewed as reflecting the different phases that the model goes through. The application of the linear analysis is valid during such a phase, but cannot be used to bridge the gap between phases (discussion of an approach to bridging this gap is contained in Eberlein 1985). A time plot of key model variables is shown in Figure 2. The model shows a definite transient stage and then goes to a steady state in which hours worked, energy level and effectiveness show continual swings. During these swings there also appear to be some shorter term oscillation that is most readily apparent in hours worked per week.

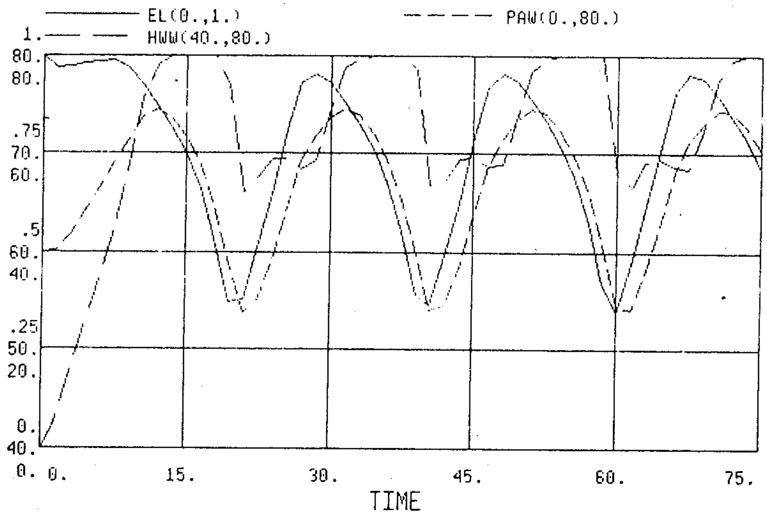


Figure 2. Energy Level (EL), Hours Worked per Week (HWW) and Perceived accomplishments per Week (PAW) in the Worker Burnout Model

In the linearization we focus on the period during which this shorter term oscillation is active. We perform a linearization at time 42. The eigenvalues of this linearization imply two oscillatory modes, one of period about 12 and one of period about

6. The one mode of these that might correspond to the behavior we are interested in getting a handle on is the 6 month oscillation[3]. This shorter term mode is important in determining the behavior of hours worked per week. In Figure 3 we show the component of hours worked per week that is attributable to this mode. Two things are to be noticed: first, thought the component goes through the major shifts that hours worked does, the timing is very different, and second, the component shows the shorter term oscillation and has timing consistent with hours worked.

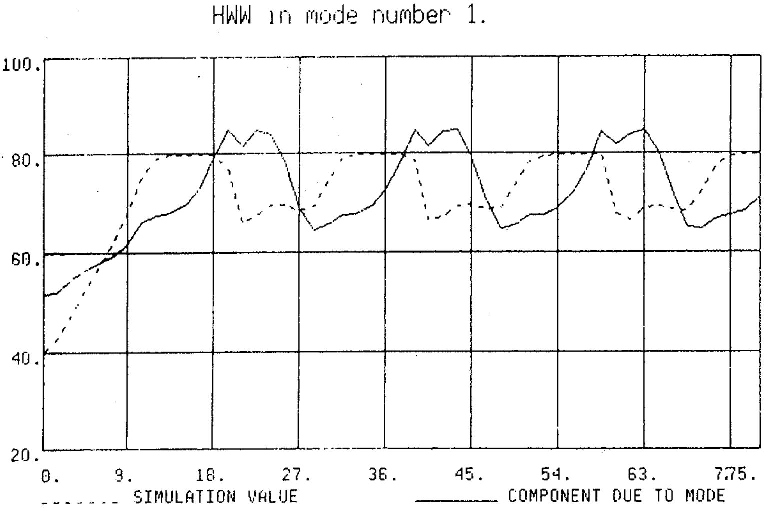


Figure 3. Behavior due to a mode for Hours Worked per Week

These findings are consistent with a mode that is causing the short term oscillations. Because the model is nonlinear the transformation of the state vectors does not end with something showing only the short term oscillation. The nonlinearity causes an environment in which the linearization we have is valid only during certain periods. During those periods the component of hours worked due to the mode of interest shows a short term oscillation. During the other periods the other dynamics dominate and the transformation of the states essentially follows

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3. Analysis of the longer oscillation does not reduce appreciably the complexity of the model, though it does suggest that expected accomplishments per week might be removed

suit. The difference in timing reflects the fact that the transformation has components from other states in the model.

If we apply the rules for state selection discussed above with a 40% cut off rule then two states, the energy level and the hours worked per week are identified as being important.[4] Based on the selection of these two states we generate a simplified model by the method outlined. The simplified model contains 10 active variables and 2 exogenous variables, both of these coming from the perceived adequacy of work which has been removed. A listing of the simplified model is contained in the appendix.

The basic feedback in the simplified model is clear in Figure 4. Although there are feedback loops involving the energy level and its renewal and depletion directly, the fundamental loop is a major negative loop through both levels. As the energy level rises so the hours worked per week rise which begins to deplete energy. This loop, with its attendant delays has the potential to cause oscillation. When the simplified model is simulated using constant values for the two exogenous inputs the results

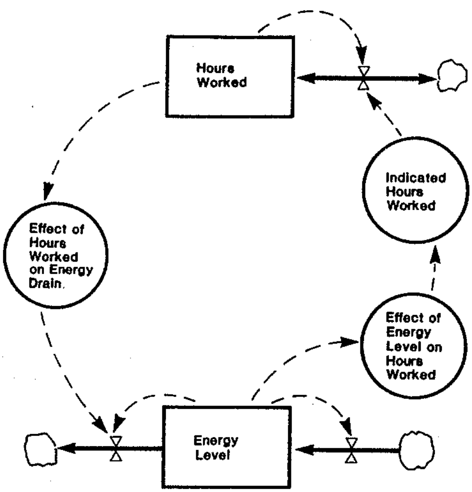


Figure 4. The simplified worker burnout model

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4. Because the mode identified is a complex mode it will necessarily require at least two states for its generation.

are a very high frequency oscillation that as is shown in Figure 5. Because we have not tried to adjust the parameters of the simplified model the frequency of oscillation does not match that of the original model, but the causes and character of the oscillation are similar.

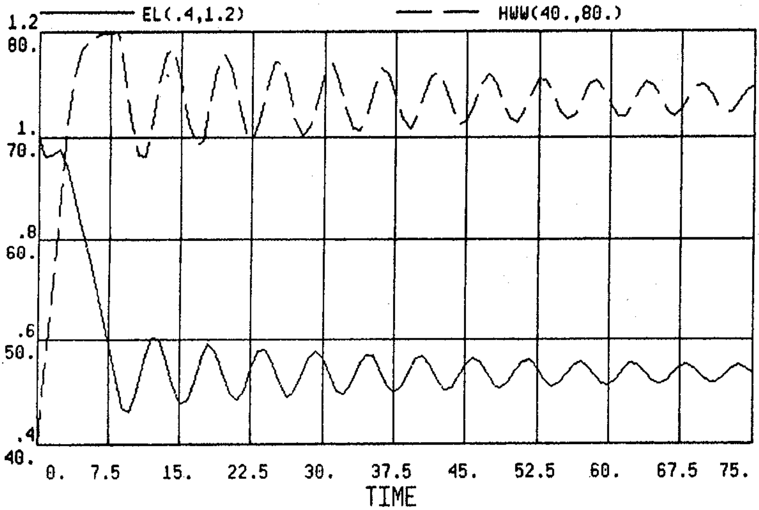


Figure 5. Simulation of Hours Worked per Week (HWW) and Energy Level (EL) in the simplified worker burnout model.

In this example we have started with a nonlinear model and shown how we can get at the structure underlying some of its behavior. This same approach will not necessarily work for all visible patterns of behavior. The longer term oscillation of the model is a more nonlinear phenomenon. Although we have seen one root implying oscillation, at most times the roots for the model are explosive (eigenvalues greater than zero), implying that the model would send things off in one direction if no nonlinearities were encountered. The linear analysis can be used to get at the causes of the explosive roots, which are important in the explanation of the longer term oscillation.

## CONCLUSIONS

In this paper we have outlined practical tools for understanding model dynamics. The tools are based on the theory of linear model analysis, but can be usefully applied to nonlinear models as the example has made clear. The key feature that distinguishes the tools described is the ease with which they can be applied. Because information is presented in the same time domain that most modelers are comfortable working with the costs of applying the tools are quite low. The tools work from a nonlinear feedback model to another smaller nonlinear feedback model. Though the second nonlinear model will not necessarily generate the original behavior modes, it does contain the essential feedback structure for doing this.

Work in linear model analysis frequently ends with the reminder that most models are nonlinear. Theoretical tools for dealing with nonlinear models are neither common nor powerful and, clearly, all additions are welcome. The emphasis on the need for a better theory of nonlinear model can, however, cause the linear model theory to be ignored. Equally as important as developing new theory is finding out when the old theory does and does not work. It is hoped that the kind of tools outlined in this paper can make that determination a reality.

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## APPENDIX: SIMPLIFIED WORKER BURNOUT MODEL

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L  HWW.K=HWW.J+(DT/TAHWW)*(IHWW.J-HWW.J)
N  HWW=HWWI          HOURS WORKED PER WEEK (HOURS/WEEK)
C  HWWI=40           HOURS WORKED PER WEEK INITIAL (HOURS/WEEK)
C  TAHWW=1           TIME TO ADJUST HOURS PER WEEK (WEEKS)
A  IHWW.K=MIN(LHWW,HWW.K*EELHW.K*EPAHW.K)
                     INDICATED HOURS WORKED PER WEEK (HOURS/WEEK)
C  LHWW=80           LIMIT ON HOURS WORKED PER WEEK (HOURS/WEEK)
A  EELHW.K=TABLE(TEELHW,EL.K,0,1,.2)
                     EFFECT OF ENERGY LEVEL ON HOURS WORKED
                       (DIMENSIONLESS)
T  TEELHW=0/.4/.7/.9/1/1
                     TABLE FOR EFFECT OF ENERGY LEVELS ON HOURS WORKED
A  EPAHW.K=1.2       EFFECT OF PERCEIVED ADEQUASCY OF HOURS WORKED
                       (DIMENSIONLESS)
L  EL.K=EL.J+DT*(ER.JK-ED.JK)
N  EL=ELI            ENERGY LEVEL (UMPH)
C  ELI=1             ENERGY LEVEL INITIAL (UMPH)
R  ER.KL=ERN*EHWER.K*EHEFR.K
                     ENERGY RECOVERY (UMPH/WEEK)
C  ERN=.3            ENERGY RECOVERY NORMAL (UMPH/WEEK)
A  EHWER.K=TABLE(TEHWER,HWW.K,0,120,20)
                     EFFECT OF HOURS WORKED ON ENERGY RECOVERY
                       (DIMENSIONLESS)
T  TEHWER=1.3/1.2/1/.7/.5/.35/.25
                     TABLE FOR EFFECT OF HOURS WORKED ON ENERGY LEVEL
A  EHEFR.K=TABHL(TEHEFR,EL.K,.8,1,.05)
                     EFFECT OF HIGH ENERGY ON FURTHER RECOVERY
                       (DIMENSIONLESS)
T  TEHEFR=1/.9/.7/.4/0
                     TABLE FOR EFFECT OF HIGH ENERGY ON FURTHER
                       RECOVERY
R  ED.KL=EDN*EPAED.K*EHWED.K*ELEFD.K
                     ENERGY DRAIN (UMPH/WEEK)
C  EDN=.06           ENERGY DRAIN NORMAL (UMPHH/WEEK)
A  EPAED.K=2         EFFECT OF PERCIEVED ADEQUACY ON ENERGY DRAIN
                       (DIMENSIONLESS)
A  EHWED.K=TABLE(TEHWED,HWW.K,0,120,20)
                     EFFECT OF HOURS WORKED ON ENERGY DRAIN
                       (DIMENSIONLESS)
T  TEHWED=.3/.6/1/.15/2/2.5/3
                     TABLE FOR EFFECT OF HOURS WORKED ON ENERGY DRAIN
A  ELEFD.K=TABHL(TELEFD,EL.K,0,.2,.05)
                     EFFECT OF LOW ENERGY ON FURTHER DEPLETION
                       (DIMENSIONLESS)
T  TELEFD=0/.4/.7/.9/1
                     TABLE FOR EFFECT OF LOW ENERGY ON FURTHER
                       DEPLETION
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