AN INVESTMENT AND PRICING PROBLEM
A SYSTEM DYNAMICS APPROACH TO THE DIFFERENTIAL GAMES

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Abstract This paper deals with the analysis of the classical investment and pricing problem of a monopoly faced with competition from substitute industries or marginal firms in the same field. The monopoly owns a finite level of a resource (ie, the stock of an exhaustible resource), whose usage is to be divided optimally over a finite planning horizon. The demand for the resource is described by a downward sloping demand curve which is affected by the measures of the competitor. The monopoly and its competitor are maximizing the present values of their net profits over the planning horizon. The problem is first formulated as a non-cooperative differential game. The necessary conditions for the Nash solution are derived.

The necessary conditions for the solutions are stated as a two-point boundary value problem which admits also an analytical solution if some simplifying assumptions are made. However, to relax these assumptions numerical solutions are computed by employing System Dynamics.

In terms of System Dynamics the two-point boundary value problems have initial states for some level variables and terminal states for some other level variables. To solve this problem with System Dynamics we have used the Newton-Raphson method. In the Newton-Raphson method two System Dynamics models are needed: one to produce a Jacobian matrix and another to produce solutions for the original problem.

1. INTRODUCTION

In this study we deal with the classical investment and pricing problem of a monopoly that attempts to maximize the present value of net profits over a finite planning horizon. The monopoly operates under demand and supply constraints. The demand constraint is represented by a demand function, the location of which, in the price-quantity space, is determined by the level of competition. Competition towards the monopoly may emerge either from substitute industries or from uncontrollable marginal firms in the same industrial field. The supply constraint states that the monopoly has only a finite level of production resource, eg. raw material, available.

The decision variable of the competitor (competitors) who also aims at maximizing the present value of its net profits, is the level of investment in production capacity. The sales rate of the competitor depends directly on the level of investments and the net profit of the competitor is decreased, of course, by the investment costs. The net profits of the monopoly as well as the net profits of the competitor are decreased by the constant marginal production costs.

The classical investment and pricing problem described above was formulated by Lieber and Barnea (1977) as a differential game problem. After some assumptions concerning the strategies of the competitor they solved the problem as an optimal control problem. Dockner and Jorgensen (1984) analysed the same problem both as a non-cooperative and as a cooperative case.

Because the structure of the competitive problem situation is dynamic the solution process turns quite naturally into differential games. Differential games are multi-phase games with continuous time. In those games, generally, the transition of n-dimensional state is described by a system of differential equations and the players may affect the motion of the state by means of control variables (vectors). When a differential game is solved according to the principles of optimal control theory, a two-point boundary value problem must be solved. In terms of System Dynamics, the two-point boundary value problems have initial states for some level variables and terminal state requirements for some other level variables.

In this study we have ended up to the numerical solution of differential games to relax some of the heavy assumptions in the analytical solutions of the original model. As a technique in the numerical and graphic solution process we used Dysmap simulation language.

In the next section not only the basic structure and solution of differential games but also the solution of the arising set of differential equations are described. This set of differential equations is solved by the Newton-Raphson method. The connection between the adopted approach and ordinal System Dynamics will be described, too.

In section 3, efforts have first been made to show the consistency of our approach. The original problem is first solved analytically and then numerically by Newton-Raphson method and then the results are compared. Next some assumptions are withdrawn and the relaxed problem is solved by our approach. When the results of the more realistic model are compared with those of the original model the value of loosening of assumptions as well as the final value of our approach can be assessed.

2. METHODOLOGY

In this section we first present the notation to be employed in the deduction of an open-loop Nash equilibrium solution to a differential game. In the second part the general differential game framework is put forward. The necessary (and sufficient) conditions for an open-loop Nash equilibrium solution are next deduced. The resulting set of differential equations, which constitutes a two-point boundary value problem, is then tackled using the Newton-Raphson method. Finally, the connection between System Dynamics and the set of differential equations will be presented and discussed.

2.1 Notation

The following notation will be employed in the presentation of the methodology

$\mathbf{x}(t)$	n-dimensional vector of the state variables characterizing the state of the system at time t
$\mathbf{u}_i(t)$	vector of the control variables of player i [1]
T	length of the planning horizon
t	time index

1) The letter i denoting the player is explicitly written only when it is of significance.

P^i	payoff function of player i
J^i	performance index of player i (objective functional)
$L^i(t)$	vector of adjoint (costate) variables of player i for each state equation
N	number of players

The vectors are all column vectors unless otherwise stated.

2.2 The general differential game set-up

The employed presentation follows that of Feichtinger and Jorgensen (1983). A similar one is used by Thepot (1983).

The development of the vector of state variables $\mathbf{x}(t)$ as a function of time is characterized by the following set of state equations

$$d\mathbf{x}(t)/dt = \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t),\mathbf{u}(t),t) \quad \mathbf{x}(0) = \mathbf{x}_0, \text{ fixed} \qquad (2.1)$$

Additionally, at least some of the state variables may have a desired terminal value that should be reached at the end of the planning horizon. This leads to the following terminal conditions

$$\mathbf{x}(T) = S(\mathbf{x}(T)) \qquad (2.2)$$

Taking into account the state equations in (2.1) and the terminal conditions in (2.2) the player i is maximizing his performance index J^i as follows

$$J^i = \int_0^T P^i(\mathbf{x}(t),\mathbf{u}(t),t)dt + S^i(\mathbf{x}(T)) \qquad (2.3)$$

The superscript i in the terminal condition part of (2.3) signifies that only those terminal conditions whose state variables are in the control of player i are included.

Furthermore, it is assumed that the control variables belong to a set of admissible controls; ie, $u_i \in U^i$ for all i. This constraint is not explicitly dealt with by including it to the problem, but only controls that satisfy this condition are considered.

It is possible to include ordinary constraints on the control variables or on the state variables (see eg Thepot (1983)). However, this will not be done in this paper.

The problem composed of maximizing (2.3) subject to (2.1) and (2.2) for player i can be formulated as maximizing the following Hamiltonian for that player

$$H^i = H^i(\mathbf{x}(t),\mathbf{u}(t),L^i(t),t) = P^i(\mathbf{x}(t),\mathbf{u}(t),t) + \mathbf{L}^T(t)\ g(\mathbf{x}(t),\mathbf{u}(t),t) \quad (2.4)$$

where the superscript T denotes the transpose of the column vector $\mathbf{L}(t)$.

In the right-hand side of (2.4) the vector multiplication contains all the state variables; ie, also those controlled by other players.

2.3 The open-loop Nash equilibrium solution

A solution to a differential game is called an open-loop solution if the vector of optimal controls $\mathbf{u}_i$ is only a function of the vector of initial states $\mathbf{x}_0$ and time t; ie, $\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}_0,t)$. Other alternatives for solutions are feedback solutions, where the vector of controls is a function of the vector of state variables and time; ie, $\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}(t),t)$. Also closed-loop no memory solutions are possible when the vector of controls contains feedback solutions which also are functions of the initials states; ie, $\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}(t),\mathbf{x}_0,t)$.

There are various solutions to differential games depending on the assumptions about the characteristics of the play. If the players are cooperating the solution concept employed is a Pareto solution. As for non-cooperative solutions if there is a precedence in the decision making order a Stackelberg solution is reached. On the other hand, if the players are aiming at security the Nash solution concept should be used. In the latter case, as long as all the players stick to their Nash solution no single player can alone raise his payoff by changing his strategy.

The necessary conditions for the open-loop Nash equilibrium solutions for player i are

$$\partial H^i/\partial \mathbf{u}_i = 0 \quad (2.5)$$

$$dL^i(t)/dt = \dot{L}^i(t) = -\partial H^i/\partial \mathbf{x} \quad (2.6)$$

$$d\mathbf{x}/dt = \dot{\mathbf{x}}(t) = g(\mathbf{x}(t),\mathbf{u}(t),t) \quad (2.7)$$

Constraints (2.5) and (2.6) are defined for all the N players separately.

In constraint (2.5) it is assumed that the solution which is looked for is in the interior of the set of controls U^i.

The behavior of the vectors of adjoint (costate) variables L(t) are defined by constraints (2.6).

Constraints (2.7) contain the original vectors of state equations as in (2.1).

Furthermore, the terminal condition of the vector of adjoint variables is defined by the transversality conditions

$$\mathbf{L}^i(T) = dS^i(\mathbf{x}(T))/d\mathbf{x} \qquad (2.8)$$

The necessary conditions (2.5) - (2.8) are also sufficient if H^i is concave in $(\mathbf{x},\mathbf{u}_i)$ for all $(\mathbf{L}^i,t)$ or if (a weaker condition) the concavity of $\max_{\mathbf{u}i(t)} H^i$ in x for all $(\mathbf{L}^i,t)$ is valid. Furthermore, S must be concave in x(T).

After solving $\mathbf{u}^*(t)$ from (2.5) as a function of $(\mathbf{x},\mathbf{L},t)$ and inserting it into (2.6) and (2.7) we have a two-point boundary value problem (ie, some state variables including the adjoint variables have initial conditions and others terminal conditions or both) composed of the differential equations in (2.6) and (2.7).

2.4 The Newton-Raphson method

To solve the set of differential equations defined by (2.6) and (2.7) it is necessary to know $\mathbf{x}(0)$ and $\mathbf{L}(0)$. However, only the first one of these initial states is known.

After the substitution mentioned above we are dealing with the following two-point boundary value problem

$$\dot{\mathbf{L}}^i(t) = -\partial H^i/\partial\mathbf{x} = \mathbf{z}^i(\mathbf{x}(t),\mathbf{L}(t),t) \qquad (2.9)$$

and

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t),\mathbf{L}(t),t) \qquad (2.10)$$

Furthermore, it is known that

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad (2.11)$$

and

$$\mathbf{L}^i(T) = dS^i(\mathbf{x}(T))/d\mathbf{x} = \mathbf{q}^i(\mathbf{x}(T)) \qquad (2.12)$$

Attention should be paid to the fact that there are differential equations like the ones in (2.9) for every player; ie, the number of players times the number of state variables.

To find the vector of initial states $L(0)$ an iterative process employing the Newton-Raphson method is put forward. Roberts and Shipman (1972) as well as Polak (1971) among others present this procedure.

Let us subsequently drop the superscript i denoting the player since all the equations are defined over the whole set of players N. A new superscript j is introduced to imply the iteration. Hence, the following notation will be adopted

L_0^j = a guess for the initial states of the adjoint variables in iteration j

$x^j(t)$ = $v(x_0, L_0^j, t)$ the value of the vector of state variables in iteration j and period t when the initial states x_0 and L_0^j are used

$L^j(t)$ = $w(x_0, L_0^j, t)$ the value of vector of adjoint variables in iteration j and period t when the initial states x_0 and L_0^j are employed

From (2.12) and the last definition above we know that

$$L^j(T) = w(x_0, L_0^j, T) = q(x(T)) = L(T) \qquad (2.13)$$

Employing the Newton-Raphson method the set of equations in (2.13) is solved by first linearizing the equation

$$w(x_0, L_0^j, T) - L(T) = w(x_0, L_0^j, T) - q(x(T)) \qquad (2.14)$$
$$= w(x_0, L_0^j, T) - q(v(x_0, L_0^j, T))$$

in the neighborhood of L_0^j. Thus,

$$w(x_0, L_0^j, T) + J\,(L_0^{j+1} - L_0^j) - L(T) = 0 \qquad (2.15)$$

where J (the Jacobian) denotes a matrix whose i,j element is

$$\partial w_i(x_0, L_0^j, T)/\partial L - \partial q_i(v(x_0, L_0^j, T))/\partial L$$

evaluated at L_0^j. It should be pointed out that in (2.15) the Jacobian J is multiplied by a column vector containing the differences between the approximations for L_0 in two consequtive iterations.

In the case of (2.15) only a first order approximation, (one containing only first derivatives) is employed, higher order terms (derivatives) could be added to improve the approximation.

On the other hand, equation (2.15) can be rewritten as

$$L^j(T) + J\,(L_0^{j+1} - L_0^j) - L(T) = 0 \qquad (2.15')$$

Solving (2.15') for L_0^{j+1} we get the following iteration sequence

$$L_0^{j+1} = L_0^j - J^{-1}\,(L_0^j(T) - L(T)) \qquad (2.16)$$

The iteration process in (2.16) only works when the inverse of the Jacobian J^{-1} is defined.

It is readily seen from (2.16) that as the approximated terminal value of L approaches the true one as the number of iterations j approaches infinity the initial state of the vector of adjoint variables approches a constant value, the solution.

In order to be able to solve the original two-point boundary value problem we must first solve the Jacobian matrix J by evaluating its components.

From above it is known that $L^j(t) = w(x_0, L_0^j, t)$. Using the following manipulation it is possible to determine a new system of differential equations, whose solution contains the derivatives from the Jacobian matrix J. Determine

$$d(\partial L^j(t)/\partial L_0^j)/dt = d(\partial w(x_0, L_0^j, t)/\partial L_0^j)/dt \qquad (2.17)$$
$$= \partial(dw(x_0, L_0^j, t)/dt)/\partial L_0^j = \partial(dL^j(t)/dt)/\partial L_0^j$$
$$= \partial z(x^j, L^j, t)/\partial L_0^j = (\partial z/\partial x)\,(\partial v/\partial L_0^j) + (\partial z/\partial L)\,(\partial w/\partial L_0^j)$$

To calculate the second part of the Jacobian we utilize the known equality $x^j(t) = v(x_0, L^j, t)$ and then determine

$$d(\partial x^j(t)/\partial L_0^j)/dt = d(\partial v(x_0, L_0^j, t)/\partial L_0^j)/dt \qquad (2.18)$$
$$= \partial(dv(x_0, L_0^j, t)/dt)/\partial L_0^j = \partial(dx^j(t)/dt)/\partial L_0^j$$
$$= \partial g(x^j, L^j, t)/\partial L_0^j = (\partial g/\partial x)\,(\partial v/\partial L_0^j) + (\partial g/\partial L)\,(\partial w/\partial L_0^j)$$

The following group of differential equations, which contains $2Nn^2$ equations (n equals the number of state equations from (2.1) and N is the number of players), is solved

$$d(\partial \mathbf{x}^j(t)/\partial \mathbf{L}_0^j)/dt = (\partial \mathbf{g}/\partial \mathbf{x})\ (\partial \mathbf{v}/\partial \mathbf{L}_0^j) + (\partial \mathbf{g}/\partial \mathbf{L})\ (\partial \mathbf{w}/\partial \mathbf{L}_0^j) \tag{2.19}$$

and

$$d(\partial \mathbf{L}^j(t)/\partial \mathbf{L}_0^j)/dt = (\partial \mathbf{z}/\partial \mathbf{x})\ (\partial \mathbf{v}/\partial \mathbf{L}_0^j) + (\partial \mathbf{z}/\partial \mathbf{L})\ (\partial \mathbf{w}/\partial \mathbf{L}_0^j) \tag{2.20}$$

with the initial conditions

$$(\partial \mathbf{x}^j(t)/\partial \mathbf{L}_0^j) = 0 \quad \text{and} \quad (\partial \mathbf{L}^j(t)/\partial \mathbf{L}_0^j) = I, \tag{2.21}$$

where I denotes an identity matrix.

The procedure to solve the problem is as follows

(i) determine from the original problem of (2.9) and (2.10) the derivatives necessary for the model in (2.19) and (2.20)

(ii) formulate the respective System Dynamics model (cf below sections 2.3 and 3) using the equations (2.19), (2.20) and (2.21)

(iii) simulate the model for T periods; ie, the length of the planning horizon

(iv) take the values for the variables in (2.19) and (2.20) in period T and insert then into the Jacobian matrix **J** making simultaneously the necessary other multiplications

(v) calculate the inverse of the Jacobian matrix $\mathbf{J}^{-1}$ using eg a Basic program

(vi) write the set of differential equations in (2.5)-(2.7) in System Dynamics

(vii) use the iteration process in (2.16) continuing with the iterations until the vector of initial states **L**(0) does not change any more

(viii) save the final solution since it is the Nash equilibrium solution

2.5 System dynamics and differential equations

So far this section has only dealt with the differential game aspect of the topic. System dynamics has only been briefly mentioned as a method in solving the two-point boundary value problem. The reasons why System Dynamics was chosen as our method is discussed next.

There is an obvious connection between differential equations and System Dynamics. This has been discussed in detail by Kivijärvi and Tuominen (1986). The state variables (both actual ones and adjoint ones) are clearly level variables in System Dynamics. The behavior of these state variables is determined by the state equations, which in the language of System Dynamics are rate variables. Auxiliary variables can be utilized to simplify the sometimes complex equations. In addition, System Dynamics provides tools modelling initial conditions, constraints on states and controls (limiting functions), jumps and delays.

3. A DYNAMIC INVESTMENT AND PRICING PROBLEM WITH OPTIMAL SOLUTIONS

In this section we analyse the classical investment and pricing problem of a monopoly faced with competition from substitute industries or marginal firms in the same field. The monopoly owns a finite level of a resource (ie, the stock of an exhaustible resource), whose usage is to be divided optimally over a finite planning horizon. The demand for the resource is described by a downward sloping demand curve which is affected by the measures of the competitor. The monopoly and its competitor are maximizing the present values of their net profits over the planning horizon. The problem is first formulated as a non-cooperative differential game. The necessary conditions for the Nash solution are derived.

3.1 Original problem with analytical solution

It is assumed that the monopoly's problem is to maximize

$$J^2 = \int_0^T (\exp(-r_2 t)(p(t)-m)D(t))dt - S_2 \qquad (3.1)$$

where

$D(t)$ = demand towards the monopoly at time t
$p(t)$ = monopoly's price at time t

m = constant marginal production cost
r_2 = discount rate
S_2 = salvage term.

The demand towards the monopoly at time t, D(t), is a linear function of the price and the competitor's sales rate, x(t). The demand constraint is

$$D(t) = a - b\,p(t) - x(t) \geq 0 \qquad (3.2)$$

where

x(t) = sales rate of the competitor at time t
= production volume of the competitor at time t
= capacity of the competitor at time t
a, b = positive constants.

The supply constraint is expressed by the level of resources available to the monopoly, y(t), at time t. The state (level) equation is given by

$$\dot{y}(t) = -D(t), \quad y(0) = y_o > 0. \qquad (3.3)$$

The problem of the price-taker competitor is to maximize

$$J^1 = \int_0^T (\exp(-r_1 t)(p(t)-k)x(t) - g(u(t)))\, dt - S_1 \qquad (3.4)$$

where

u(t) = competitor's investment in capacity at time t
g(u(t)) = investment cost function (convex)
r_1 = discount rate
k = constant marginal production cost
S_1 = salvage term.

The sales rate of the competitor (production rate, resource level) is given by the second state equation

$$\dot{x}(t) = -\mu\, x(t) + u(t), \quad x(0) = x_o \geq 0, \qquad (3.5)$$

where

μ = depreciation (deterioration) rate of capacity.

The state variables and the demand variable are constrained by $x(t) \geq 0$, $y(t) \geq 0$, $D(t) \geq 0$. A two-person nonzero-sum differential game is defined by equations (3.1)-(3.5). In the original model (see Lieber - Barnea (1977) and Dockner - Jorgensen (1984)) following five assumptions are made:

(1) $t \in (0,T)$ for both players
(2) $g(u) = \beta u^2$, $\beta > 0$ and constant
(3) $r_1 = r_2 = 0$
(4) $\mu = 0$
(5) $S_1 = S_2 = 0$.

The current value Hamiltonians for the original problem are (t is dropped out for brevity)

$$H^1 = (p-k)x - \beta u^2 + L^1_1 u + L^1_2(-a+bp+x) \qquad (3.6)$$

$$H^2 = (p-m)(a-bp-x) + L^2_1 u + L^2_2(-a+bp+x), \qquad (3.7)$$

where

L^i_j = current value adjoint variables, i,j = 1,2.

Necessary and sufficient conditions for the open-loop Nash equilibrium solution with (3.3) and (3.5) are

$$\partial H^1 / \partial u = -2\beta u + L^1_1 = 0 \qquad (3.8)$$

$$\dot{L}^1_1 = -\partial H^1 / \partial x = k - p - L^1_2 \qquad (3.9)$$

$$\dot{L}^1_2 = -\partial H^1 / \partial y = 0 \qquad (3.10)$$

$$\partial H^2 / \partial p = -2bp + a + bm - x - b L^2_2 = 0 \qquad (3.11)$$

$$\dot{L}^2_1 = -\partial H^2 / \partial x = p - m - L^2_2 \qquad (3.12)$$

$$\dot{L}^2_2 = -\partial H^2 / \partial y = 0 \qquad (3.13)$$

Applying the transversality conditions the control variables are solved from (3.8) and (3.11), respectively, as follows:

$$u^* = L^1_1 / 2\beta \qquad (3.14)$$

$$p^* = m/2 + a/2b - x/2b. \qquad (3.15)$$

The necessary and sufficient conditions can be presented in the form of the following two-point boundary value problem:

$$\dot{x} = L^1_1 / 2\beta \qquad x(0) = x_o \qquad (3.16)$$

$$\dot{y} = (bm - a)/2 + x/2 \qquad y(0) = y_o \qquad (3.17)$$

$$\dot{L}^1_1 = k + x/2b - m/2 - a/2b \qquad L^1_1(T) = 0 \qquad (3.18)$$

$$\dot{L}^2_2 = a/2b - m/2 - x/2b \qquad L^2_2(T) = 0 \qquad (3.19)$$

If we define $K = 2\sqrt{(\beta\ b)}$ then it is easy to obtain the following analytically optimal solution of the problem:

$$x^* = a+bm-2kb+(x_0-a-bm+2kb)(\cosh t/K - \tanh T/K \sinh t/K) \qquad (3.20)$$

$$u^* = ((x_0-a-bm+2kb)/K)(\sinh t/K - \tanh T/K \cosh t/K) \qquad (3.21)$$

$$L^1_1 = \sqrt{(\beta/b)}(x^0-a-bm+2kb)(\sinh t/K - \tanh T/K \cosh t/K) \qquad (3.22)$$

$$p^* = k + ((x_0-a-bm+2kb)/2b)(\tanh T/K \sinh t/K - \cosh t/K) \qquad (3.23)$$

$$D^* = (k-m)b + (1/2)(x_0-a-bm+2kb)(\tanh T/K \sinh t/K - \cosh t/K) \qquad (3.24)$$

A program list in Dysmap language equivalent to the analytical solution (3.20) - (3.24) is given in Appendix 1A and the respective numerical solution in Appendix 1B. The numerical values of the parameters appear in the program listing.

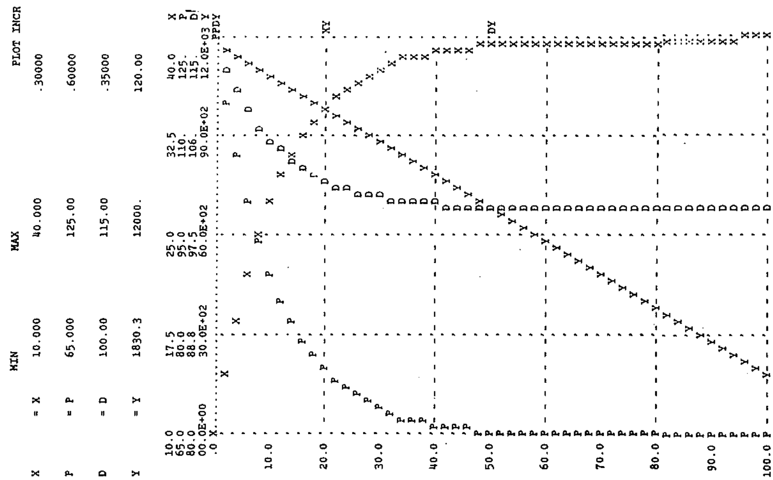

Figure 1 Analytical solution of original problem

The graphic solution of the problem is shown in the Figure 1. The results indicate that the monopoly decreases the price, p, and the competitor increases its production capacity, x, by investments. The net effect of the decreasing price and the competitor's increasing sales is the decreasing demand, D. The level of the monopoly's resource, y, is decreasing, but at decreasing rate.

3.2 Original problem with Newton-Raphson solution

To evaluate the appropriateness and effectiveness of the Newton-Raphson method we solved the original game-problem also with the Newton-Rapson method (see eg. Polak (1971)). As stated in Section 2 for the Newton-Raphson method we need an inverse of the Jacobian matrix. For this purpose we start with the original problem (3.16) - (3.19) and redefine the equations as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{L} + \mathbf{C}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \qquad (3.25)$$

$$\dot{\mathbf{L}} = \mathbf{D}\,\mathbf{x} + \mathbf{E}, \qquad \mathbf{L}(T) = \mathbf{L}_T, \qquad (3.26)$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} L_1^1 \\ L_2^2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ .5 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1/(2\ ) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 \\ (b/m - a)/2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1/(2b) & 0 \\ -1/(2b) & 0 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} k - a/2b \\ a/(2b) - m/2 \end{bmatrix}$$

$$\mathbf{x}_0 = \begin{bmatrix} x_o \\ y_o \end{bmatrix} \qquad \mathbf{L}_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To obtain the Jacobian matrix we must solve the following system of differential equations:

$$d(\partial \mathbf{x}^j(t)/\partial \mathbf{L}_0^j)/dt = \mathbf{A}(\partial \mathbf{x}^j/\partial \mathbf{L}_0^j) + \mathbf{B}(\partial \mathbf{L}^j/\partial \mathbf{L}_0^j) \qquad (3.27)$$

$$d(\partial \mathbf{L}^j(t)/\partial \mathbf{L}_0^j)/dt = \mathbf{D}(\partial \mathbf{x}^j/\partial \mathbf{L}_0^j) + \mathbf{0}(\partial \mathbf{L}^j/\partial \mathbf{L}_0^j) \qquad (3.28)$$

with boundary conditions:

$$(\partial x^j(t_0)/\partial L_0^j) = 0, \qquad (\partial L^j(t_0)/\partial L_0^j) = I \qquad (3.29)$$

where I = identity matrix.

For solving this system of differential equations an auxiliary Dysmap-model was formulated. In the model following symbols are used:

$Z1 = \partial x / \partial L_{01}^1$ $\qquad S1 = \partial L_1^1 / \partial L_{01}^1$

$Z2 = \partial y / \partial L_{01}^1$ $\qquad S2 = \partial L_1^2 / \partial L_{01}^1$

$Z3 = \partial x / \partial L_{02}^2$ $\qquad S3 = \partial L_1^1 / \partial L_{02}^2$

$Z4 = \partial y / \partial L_{02}^2$ $\qquad S4 = \partial L_1^2 / \partial L_{02}^2$

The equivalent program in Dysmap language is provided in Appendix 2A and the respective numerical solution in Appendix 2B. The results in Appendix 2B give the Jacobian

$$J = \begin{bmatrix} 3365.0 & 0 \\ -33646.6 & 1 \end{bmatrix} \qquad (3.30)$$

and its inverse

$$J^{-1} = \begin{bmatrix} 2.971238E-04 & 0 \\ 9.997028E-01 & 1 \end{bmatrix} \qquad (3.31)$$

Now it is possible to formulate a Dysmap model equivalent to the original problem (equations (3.16) - (3.19)). The listing of the model is provided in Appendix 2C. In the model, as a starting point for iterations, 100 was assigned for L1 and L2. When the model was solved, Appendix 2C1, the terminal value of L1 = 19.377E+04 and L2 = -19.107E+04 although they should be 0.

In the next iteration new initial values for the adjoint variables, L1 and L2, are computed by using the Newton-Raphson method (equation 2.16). New values are 42.426314 and -2542.4271. Results are given in Appendix 2C2. Now the terminal values of L1 and L2 are quite close to 0, but one more iteration is still computed. The final results are presented in Appendix 2C3, where the terminal value of L1 = -0.011 and L2 = 0.012. These values are sufficiently close to zeros and accepted.

If the results in the Appendix 2C3 are compared with the analytical results presented in Appendix 1B it is easy to notice that there are only insignificant deviations between the results in the end. However, it should be pointed out that the price variable p^* in the Newton-Raphson approach is the myopic optimal price. Thus, the Newton-Raphson method to find the Nash equilibrium seems to be valid. In the next section, the heavy assumptions of the original problem are relaxed and the new problem is solved by Newton-Raphson method.

3.3 A more realistic problem with Newton-Raphson solution

Next, the assumptions of the original model are changed as follows:

(1) $t \in (0,T)$ for both players
(2) $g(u) = \beta u^2$, $\beta > 0$ and constant
(3) $r_1 = r_2 > 0$
(4) $\mu > 0$
(5) $S_1 = 0$, $S_2 = (y(T) - \rho y_o)^2$, $1 > \rho > 0$.

These changes in assumptions indicate that the net profits are discounted by the same positive discount rate, the production capacity of the competitor is decreased at μ rate, and that there is a requirement for the terminal value of the resource of the monopol.

Now, the necessary and sufficient conditions for optimal pricing and investment decisions are

$$\dot{x} = - \mu x + L_1^1 / 2 \beta \qquad (3.32)$$

$$\dot{y} = (bm-a)/2+x/2+b L_2^2/2 \qquad (3.33)$$

$$\dot{L}_1^1 = (r + \mu) L_1^1 - L_2^1 + k - p \qquad (3.34)$$

$$\dot{L}_2^1 = r L_2^1 \qquad (3.35)$$

$$\dot{L}_1^2 = (r + \mu) L_1^2 - L_2^2 - m - p \qquad (3.36)$$

$$\dot{L}_2^2 = r L_2^2 \qquad (3.37)$$

when the following control variables are used:

$$u^* = L_1^1 / 2 \beta \qquad (3.38)$$

$$p^* = a / 2b - x / 2b + m/2 + L_2^2 / 2 \qquad (3.39)$$

An auxiliary Dysmap-model for solving the Jacobian matrix is listed in Appendix 3A and its solution in Appendix 3B. The variables are defined with the same principle as in Appendix 2A. The inverse of the Jacobian is

$$J^{-1} = \begin{bmatrix} -1.08857E-02 & 9.08553E-01 & 0 & 2.56182E-03 \\ 0 & 1.35505E-01 & 0 & 0 \\ -4.26106E-02 & -1.98201E-01 & 6.79071E-03 & 1.15831E-02 \\ -1.90043E-03 & -1.12865E-01 & 0 & 4.45893E-04 \end{bmatrix} \quad (3.40)$$

A model in Dysmap language equivalent to the revised problem (equations (3.31) - (3.38)) is listed in Appendix 4A.

Four iterations were computed. The initial and terminal values of the adjoint variables are given in Table 1 and the other results of the iteration process in Appendix 4B. Notice, that in the iteration process the terminal value requirement must be included in equation 2.16

Iter	L11	L12	L21	L22
0	50 (-25.720E+06)	50 (368.99)	50 (25.341E+06)	50 (368.99)
1	41.419 (703.70)	0 0	-514.65 (-343.86)	.66111 (4.8798)
2	48.336 (3.1679)	0 0	-485.69 (7.4281)	1.8690 (13.793)
3	48.372 (.30981)	0 0	-485.60 (-.19321)	1.8754 (13.840)
4	48.372 (44.435E-03)	0 0	-485.60 (-75.728E-03)	1.8754 (13.840)

Table 1: Initial and (terminal) values of the adjoint variables

The terminal values of the adjoint variables L11, L12 and L21 are relatively close to zero, as they should be according to the assumptions. The value of L22 is 13.840 as indicated by the terminal state requirement.

The graphic solution of the revised problem is shown in Figure 2, which should be compared with Figure 1.

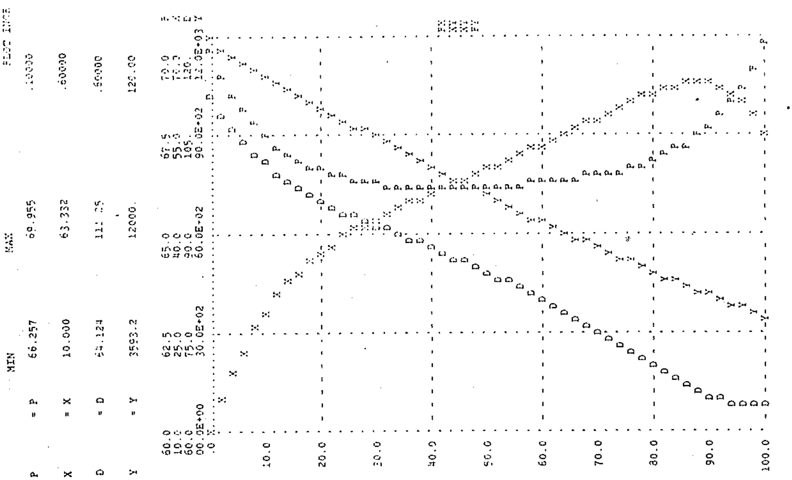

Figure 2 Newton-Raphson solution of revised problem

Clearly, there are significant differences between the patters of solutions. Even in this case the level of the monopoly's resource, y, is decreasing. However, its final value is at a much higher level than in the original solution. The shape of the monopoly's pricing behaviour is interesting. First, the price, p, is decreasing but turns then to increase. On the contrary, the level of competitor's capacity is first decreasing but at the end of the period it is increasing again.

All these drastic changes in the behaviour in the solution are possible due to the changes in the assumptions of the model. And because these changes are due to the adopted numerical approach its contribution to investment and pricing problem is remarkable.

4. CONCLUSIONS

In this paper we have dealt with an investment and pricing problem of a monopoly faced with competition employing System Dynamics to solve this differential game problem. After the introduction and general characterization of the problem in section 1 the methodology was presented in section 2. That section also contained a brief description of the differential game setting in general and a deduction of the necessary conditions for an open-loop Nash equilibrium. The connections between differential equations and System Dynamics were established. Section 3 contained a mathematical model which had been solved analytically in a previous study. This model was then solved with System Dynamics which resulted in the same solution. However, the original model contained rather heavy simplifying assumptions, some of which were relaxed in the study. Our methodology was then utilized in this more realistic model and the model was successfully solved in four iterations.

The original model employed was a very simple one with only linear functions in the resulting two-point boundary value problem. The proposed methodology is by no means restricted to these assumptions. An attempt to prove this was made by formulating the methodology in as general a way as possible. As a generalization nonlinear state equations will be dealt with in a future study where the Finnish forest sector is analyzed.

In this study only Nash solutions for the problem were determined. There are also other possible equilibrium solutions such as Pareto or Stackelberg equilibriums. Although the optimality conditions for these are different they all the same constitute a two-point boundary value problem which be can solved with the methodolgy presented in this paper.

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APPENDIX 1A. A PROGRAM LIST EQUIVALENT TO THE ANALYTICAL SOLUTION OF THE ORIGINAL PROBLEM

```
* ANALYTICALLY OPTIMAL
NOTE ----------------------------------------------------------
NOTE HEAVY ASSUMPTIONS
NOTE ----------------------------------------------------------
A X.K=A+B*M-2*K*B+C*(COSH.K-TANH.K*SINH.K)         COMPETITOR'S SALES
A U.K=(C/K)*(SINH.K-TANH.K*COSH.K)            COMPETITOR'S INVESTMENT
A L.K=SQRT(BE/B)*C*(SINH.K-TANH.K*COSH.K)            ADJOINT VARIABLE
A P.K=K+(C/2*B)*(TANH.K*SINH.K-COSH.K)                          PRICE
A D.K=(K-M)*B+0.5*C*(TANH.K*SINH.K-COSH.K)                     DEMAND
A Y.K=YI+B*(M-K)*TIME.K+(KK/2)*C*(SINH.K+TANH.K*(1-COSH.K))  RESOURCE
NOTE ----------------------------------------------------------
C A=400                PARAMETER OF DEMAND FUNCTION
C B=4                  PARAMETER OF DEMAND FUNCTION
C M=40                 MARGINAL PRODUCTION COST (MONOPOL)
C K=65                 MARGINAL PRODUCTION COST (COMPETITOR)
C BE=8                 PARAMETER OF INVESTMENT COST FUNCTION
C XI=10                INITIAL SALES RATE
C YI=12000             INITIAL RESOURCE
NOTE ----------------------------------------------------------
N KK=2*(SQRT(BE*B))  CAPITAL K
N C=XI-A-B*M+2*K*B  CONSTANT
NOTE ----------------------------------------------------------
A COSH.K=((EXP(TIME.K/KK))+(EXP(-(TIME.K/KK))))/2
A SINH.K=((EXP(TIME.K/KK))-(EXP(-(TIME.K/KK))))/2
A TANH.K=((EXP(LENGTH/KK))-(EXP(-(LENGTH/KK))))/
X ((EXP(LENGTH/KK))+(EXP(-(LENGTH/KK))))
NOTE ----------------------------------------------------------
C LENGTH=100
C DT=0.0625
C PRTPER=5
C PLTPER=2
NOTE ----------------------------------------------------
PRINT 1)P,U/2)D/3)Y,X/4)L
PLOT X=X(10,40)/P=P(65,125)/D=D(80,115)/Y=Y(0,12000)
RUN ANALYTICAL SOLUTION
+
```

APPENDIX 1B. NUMERICAL RESULTS OF THE ANALYTICAL SOLUTION

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PAGE 1

ANALYTICALLY OPTIMAL ANALYTICAL SOLUTI

TIME	P U	D	Y X	L
00.00E+00	125.00 .46154	115.00	12000. 10.000	42.426
5.000	103.57 .29667	109.64	11439. 20.716	27.271
10.00	89.791 .19070	106.20	10900. 27.605	17.530
15.00	80.935 .12258	103.98	10375. 32.032	11.268
20.00	75.243 78.791E-03	102.56	9859.3 34.879	7.2428
25.00	71.584 50.646E-03	101.65	9348.9 36.708	4.6556
30.00	69.232 32.554E-03	101.06	8842.3 37.884	2.9925
35.00	67.720 20.925E-03	100.68	8338.0 38.640	1.9235
40.00	66.749 13.451E-03	100.44	7835.2 39.126	1.2365
45.00	66.124 86.447E-04	100.28	7333.5 39.438	.79465
50.00	65.722 55.565E-04	100.18	6832.3 39.639	.51078
55.00	65.465 35.776E-04	100.12	6331.6 39.767	.32887
60.00	65.298 22.959E-04	100.07	5831.1 39.851	.21104
65.00	65.190 14.648E-04	100.05	5330.8 39.905	.13465
70.00	65.125 95.778E-05	100.03	4830.6 39.938	88.043E-03
75.00	65.081 61.974E-05	100.02	4330.5 39.960	56.969E-03
80.00	65.044 33.804E-05	100.01	3830.4 39.978	31.074E-03
85.00	65.029 22.536E-05	100.01	3330.4 39.985	20.716E-03
90.00	65.029 22.536E-05	100.01	2830.4 39.985	20.716E-03
95.00	65.000 00.000E+00	100.00	2330.3 40.000	00.000E+00
100.0	65.000 00.000E+00	100.00	1830.3 40.000	00.000E+00

APPENDIX 2A. CALCULATION OF JACOBIAN FOR THE ORIGINAL PROBLEM

```
* CALCULATION OF JACOBIAN
NOTE ------------------------------------------------------------
R Z1D.KL=(1/(2*BE))*S1.K
R Z2D.KL=0.5*Z1.K
R Z3D.KL=(1/(2*BE))*S3.K
R Z4D.KL=0.5*Z3.K
NOTE ------------------------------------------------------------
R S1D.KL=(1/(2*B))*Z1.K
R S2D.KL=-(1/(2*B))*Z1.K
R S3D.KL=(1/(2*B))*Z3.K
R S4D.KL=-(1/(2*B))*Z3.K
NOTE ------------------------------------------------------------
L Z1.K=Z1.J+DT*Z1D.JK
L Z2.K=Z2.J+DT*Z2D.JK
L Z3.K=Z3.J+DT*Z3D.JK
L Z4.K=Z4.J+DT*Z4D.JK
NOTE ------------------------------------------------------------
L S1.K=S1.J+DT*S1D.JK
L S2.K=S2.J+DT*S2D.JK
L S3.K=S3.J+DT*S3D.JK
L S4.K=S4.J+DT*S4D.JK
NOTE ------------------------------------------------------------
N Z1=0
N Z2=0
N Z3=0
N Z4=0
NOTE ------------------------------------------------------------
N S1=1
N S2=0
N S3=0
N S4=1
NOTE ------------------------------------------------------------
C B=4
C BE=8
NOTE ------------------------------------------------------------
C DT=0.0625
C LENGTH=100
C PRTPER=50
PRINT 1)Z1,Z2/2)Z3,Z4/3)S1.S2/4)S3,S4
RUN JACOBIAN
+
```

APPENDIX 2B. NUMERICAL RESULTS

CALCULATION OF JACOBIAN JACOBIAN

TIME	Z1 / Z2	Z3 / Z4	S1 / S2	S3 / S4
00.00E+00	00.000E+00	00.000E+00	1.0000	00.000E+00
	00.000E+00	00.000E+00	00.000E+00	1.0000
50.00	29.003	00.000E+00	41.028	00.000E+00
	160.11	00.000E+00	-40.028	1.0000
100.0	2379.8	00.000E+00	3365.6	00.000E+00
	13458.	00.000E+00	-3364.6	1.0000

APPENDIX 2C. NEWTON-RAPHSON SOLUTION OF THE ORIGINAL PROBLEM

```
* NEWTON RAPHSON APPROACH
NOTE
NOTE HEAVY ASSUMPTIONS
NOTE
NOTE DECISION VARIABLES
A P.K=M/2+A/(2*B)-X.K/(2*B)
A U.K=L1.K/(2*BE)
NOTE RESPECTIVE DEMAND
A D.K=A-B*P.K-X.K
NOTE STATE TRANSITION VARIABLES
R XD.KL=L1.K/(2*BE)
R YD.KL=(B*M-A)/2+X.K/2
NOTE ADJOINT TRANSITION VARIABLES
R L1D.KL=K+X.K/(2*B)-M/2-A/(2*B)
R L2D.KL=A/(2*B)-M/2-X.K/(2*B)
NOTE STATE VARIABLES
L X.K=X.J+DT*XD.JK
L Y.K=Y.J+DT*YD.JK
NOTE ADJOINT VARIABLES
L L1.K=L1.J+DT*L1D.JK
L L2.K=L2.J+DT*L2D.JK
NOTE INITIAL STATES
C ZERO=0.0
N X=XI
N Y=YI
N L1=100.
N L2=100.
NOTE PARAMETERS
C A=400          PARAMETER OF DEMAND FUNCTION
C B=4            PARAMETER OF DEMAND FUNCTION
C M=40           MARGINAL PRODUCTION COST (MONOPOL)
C K=65           MARGINAL PRODUCTION COST (COMPETITOR)
C BE=8           PARAMETER OF INVESTMENT COST FUNCTION
C XI=10          INITIAL SALES RATE
C YI=12000       INITIAL RESOURCE
NOTE -------------------------------------------
C LENGTH=100
C DT=0.0625
C PRTPER=50
C PLTPER=5
PRINT 1)L1,L2/2)P,U/3)D/4)X,Y
RUN
+
```

NEWTON RAPHSON APPROACH

TIME	L1 / L2	P / U	D	X / Y
00.00E+00	100.00	68.750	115.00	10.000
	100.00	6.2500		12000.
50.00	2362.6	-143.68	-734.72	1709.4
	-912.63	147.66		16051.
100.0	19.377E+04	-17062.	-68408.	13.706E+04
	-19.107E+04	12111.		77.668E+04

NEWTON RAPHSON APPROACH

TIME	L1 / L2	P / U	D	X / Y
00.00E+00	42.426	68.750	115.00	10.000
	-2542.4	2.6516		12000.
50.00	.50211	65.045	100.18	39.641
	-1250.5	31.382E-03		6832.3
100.0	-.20358	65.019	100.08	39.848
	.20020	-12.724E-03		1829.5

NEWTON RAPHSON APPROACH

TIME	L1 / L2	P / U	D	X / Y
00.00E+00	42.426	68.750	115.00	10.000
	-2542.4	2.6516		12000.
50.00	.50446	65.045	100.18	39.643
	-1250.5	31.529E-03		6832.3
100.0	-11.500E-03	65.002	100.01	39.983
	12.220E-03	-71.877E-05		1830.3

APPENDIX 3A. REVISED MODEL / CALCULATION OF JACOBIAN

```
* REVISED MODEL / CALCULATION OF JACOBIAN
NOTE ----------------------------------------------
R Z1D.KL=-MYY*Z1.K+(1/(2*BE))*S1.K
R Z2D.KL=0.5*Z1.K +(B/2)*S4.K
R Z3D.KL=-MYY*Z3.K+(1/(2*BE))*S5.K
R Z4D.KL=0.5*Z3.K +(B/2)*S8.K
R Z5D.KL=-MYY*Z5.K+(1/(2*BE))*S9.K
R Z6D.KL=0.5*Z5.K+(B/2)*S12.K
R Z7D.KL=-MYY*Z7.K+(1/(2*BE))*S13.K
R Z8D.KL=0.5*Z7.K+(B/2)*S16.K
NOTE ----------------------------------------------
R S1D.KL=(1/(2*B))*Z1.K+(R+MYY)*S1.K-S2.K-0.5*S4.K
R S2D.KL=R*S2.K
R S3D.KL=-(1/(2*B))*Z1.K+(R+MYY)*S3.K-0.5*S4.K
R S4D.KL=R*S4.K
R S5D.KL=(1/(2*B))*Z3.K+(R+MYY)*S5.K-S6.K-0.5*S8.K
R S6D.KL=R*S6.K
R S7D.KL=-(1/(2*B))*Z3.K+(R+MYY)*S7.K-0.5*S8.K
R S8D.KL=R*S8.K
R S9D.KL=(1/(2*B))*Z5.K+(R+MYY)*S9.K-S10.K-0.5*S12.K
R S10D.KL=R*S10.K
R S11D.KL=-(1/(2*B))*Z5.K+(R+MYY)*S11.K-0.5*S12.K
R S12D.KL=R*S12.K
R S13D.KL=(1/(2*B))*Z7.K+(R+MYY)*S13.K-S14.K-0.5*S16.K
R S14D.KL=R*S14.K
R S15D.KL=-(1/(2*B))*Z7.K+(R+MYY)*S15.K-0.5*S16.K
R S16D.KL=R*S16.K
NOTE ----------------------------------------------
L Z1.K=Z1.J+DT*Z1D.JK
L Z2.K=Z2.J+DT*Z2D.JK
L Z3.K=Z3.J+DT*Z3D.JK
L Z4.K=Z4.J+DT*Z4D.JK
L Z5.K=Z5.J+DT*Z5D.JK
L Z6.K=Z6.J+DT*Z6D.JK
L Z7.K=Z7.J+DT*Z7D.JK
L Z8.K=Z8.J+DT*Z8D.JK
NOTE ----------------------------------------------
L S1.K=S1.J+DT*S1D.JK
L S2.K=S2.J+DT*S2D.JK
L S3.K=S3.J+DT*S3D.JK
L S4.K=S4.J+DT*S4D.JK
L S5.K=S5.J+DT*S5D.JK
L S6.K=S6.J+DT*S6D.JK
L S7.K=S7.J+DT*S7D.JK
L S8.K=S8.J+DT*S8D.JK
L S9.K=S9.J+DT*S9D.JK
L S10.K=S10.J+DT*S10D.JK
L S11.K=S11.J+DT*S11D.JK
L S12.K=S12.J+DT*S12D.JK
L S13.K=S13.J+DT*S13D.JK
L S14.K=S14.J+DT*S14D.JK
L S15.K=S15.J+DT*S15D.JK
L S16.K=S16.J+DT*S16D.JK
NOTE ----------------------------------------------
N Z1=0
N Z2=0
N Z3=0
N Z4=0
N Z5=0
N Z6=0
N Z7=0
N Z8=0
NOTE ----------------------------------------------
N S1=1
N S2=0
N S3=0
N S4=0
N S5=0
N S6=1
N S7=0
N S8=0
N S9=0
N S10=0
N S11=1
N S12=0
N S13=0
N S14=0
N S15=0
N S16=1
C B=4
C BE=8
C MYY=0.03
C R=0.02
NOTE ----------------------------------------------
C DT=0.0625
C LENGTH=100
C PRTPER=50
PRINT 1)Z1,Z2,Z3.Z4,Z5,Z6,Z7,Z8
PRINT 2)S1,S2,S3,S4.S5.S6.S7.S8/3)S9,S10,S11,S12,S13,S14,S15,S16
RUN JACOBI
+
```

APPENDIX 3B. NUMERICAL RESULTS FOR JACOBIAN / REVISED PROBLEM

REVISED MODEL / CALCULATION OF JACOBIAN JACOBI

TIME	Z1 Z2 Z3 Z4 Z5 Z6 Z7 Z8	S1 S2 S3 S4 S5 S6 S7 S8	S9 S10 S11 S12 S13 S14 S15 S16
00.00E+00	00.000E+00	1.0000	00.000E+00
	00.000E+00	00.000E+00	00.000E+00
	00.000E+00	00.000E+00	1.0000
	00.000E+00	00.000E+00	00.000E+00
	00.000E+00	00.000E+00	00.000E+00
	00.000E+00	1.0000	00.000E+00
	00.000E+00	00.000E+00	00.000E+00
	00.000E+00	00.000E+00	1.0000
50.00	66.696	146.23	00.000E+00
	308.30	00.000E+00	00.000E+00
	-748.32	-134.10	12.135
	-3293.0	00.000E+00	00.000E+00
	00.000E+00	-1665.8	-832.90
	00.000E+00	2.7166	00.000E+00
	-374.16	1351.8	518.95
	-1474.8	00.000E+00	2.7166
100.0	13812.	30280.	00.000E+00
	64528.	00.000E+00	00.000E+00
	-15.868E+04	-30133.	147.26
	-74.052E+04	00.000E+00	00.000E+00
	00.000E+00	-34.793E+04	-17.397E+04
	00.000E+00	7.3798	00.000E+00
	-79338.	34.327E+04	16.930E+04
	-36.962E+04	00.000E+00	7.3798

APPENDIX 4A. NEWTON-RAPHSON SOLUTION / REVISED MODEL

```
* NEWTON RAPHSON APPROACH
NOTE MORE REALISTIC ASSUMPTIONS
NOTE
NOTE DECISION VARIABLES
A P.K=M/2+A/(2*B)-X.K/(2*B)+L22.K/2
A U.K=L11.K/(2*BE)
NOTE RESPECTIVE DEMAND
A D.K=A-B*P.K-X.K
NOTE STATE TRANSITION VARIABLES
R XD.KL=-MYY*X.K+L11.K/(2*BE)
R YD.KL=(B*M-A)/2+X.K/2+(B*L22.K)/2
NOTE ADJOINT TRANSITION VARIABLES
R L11D.KL=(R+MYY)*L11.K-L12.K+K+X.K/(2*B)-M/2-A/(2*B)-L22.K/2
R L12D.KL=R*L12.K
R L21D.KL=(R+MYY)*L21.K-L22.K/2+A/(2*B)-M/2-X.K/(2*B)
R L22D.KL=R*L22.K
NOTE STATE VARIABLES
L X.K=X.J+DT*XD.JK
L Y.K=Y.J+DT*YD.JK
NOTE ADJOINT VARIABLES
L L11.K=L11.J+DT*L11D.JK
L L12.K=L12.J+DT*L12D.JK
L L21.K=L21.J+DT*L21D.JK
L L22.K=L22.J+DT*L22D.JK
NOTE INITIAL STATES
N X=XI
N Y=YI
N L11=IL1
N L12=IL2
N L21=IL3
N L22=IL4
NOTE INITIAL VALUES OF ADJOINT VARIABLES
C IL1=48.37228117
C IL2=5.2304E-24
C IL3=-485.5970190
C IL4=1.875395286
NOTE PARAMETERS
C A=400          PARAMETER OF DEMAND FUNCTION
C B=4            PARAMETER OF DEMAND FUNCTION
C M=40           MARGINAL PRODUCTION COST (MONOPOL)
C K=65           MARGINAL PRODUCTION COST (COMPETITOR)
C BE=8           PARAMETER OF INVESTMENT COST FUNCTION
C MYY=0.03       DETERIORATION OF CAPACITY
C R=0.02         DISCOUNT RATE
C XI=10          INITIAL SALES RATE
C YI=12000       INITIAL RESOURCE
NOTE -----------------------------------------------
C LENGTH=100
C DT=0.0625
C PRTPER=10
C PLTPER=2
PRINT 1)L11,L12,L21,L22/2)P,U,D/3)X,Y
PLOT P=P(60,70)/X=X(10,70)/D=D(60,120)/Y=Y(0,12000)
RUN FINAL ITERATION
+
```

APPENDIX 4B. NUMERICAL RESULTS / REVISED PROBLEM

NEWTON RAPHSON APPROACH — ITERATION 0

TIME	L11 L12 L21 L22	P U D	X Y
00.00E+00	50.000 50.000 50.000 50.000	93.750 3.1250 15.000	10.000 12000.
50.00	-12.310E+04 135.83 98491. 135.83	7045.8 -7694.0 27480.	-55263. -22.786E+04
100.0	-25.720E+06 368.99 25.341E+06 368.99	14.664E+05 -16.075E+05 58.641E+05	-11.729E+06 -54.705E+06

NEWTON RAPHSON APPROACH — ITERATION 1

TIME	L11 L12 L21 L22	P U D	X Y
00.00E+00	41.419 10.000E-09 -514.65 .66111	69.081 2.5887 113.68	10.000 12000.
50.00	24.484 27.166E-09 -407.19 1.7960	65.805 1.5303 96.038	40.740 6908.3
100.0	703.40 73.799E-09 -343.86 4.8789	27.758 43.963 -68.485	357.46 3742.1

NEWTON RAPHSON APPROACH — ITERATION 2

TIME	L11 L12 L21 L22	P U D	X Y
00.00E+00	48.336 -13.550E-14 -485.69 1.8690	69.684 3.0210 111.26	10.000 12000.
50.00	29.843 -36.810E-14 -356.41 5.0772	66.276 1.8652 84.793	50.105 7259.2
100.0	3.1679 -99.996E-14 7.4281 13.793	69.765 .19799 63.893	57.042 3592.7

NEWTON RAPHSON APPROACH — ITERATION 3

TIME	L11 L12 L21 L22	P U D	X Y
00.00E+00	48.372 -39.300E-20 -485.60 1.8754	69.688 3.0233 111.25	10.000 12000.
50.00	29.875 -10.676E-19 -356.86 5.0946	66.278 1.8672 84.733	50.156 7261.0
100.0	.30981 -29.003E-19 -.19321 13.840	69.940 19.363E-03 64.401	55.839 3593.7

APPENDIX 4B. Continued

NEWTON RAPHSON APPROACH FINAL ITERATION

TIME	L11 L12 L21 L22	P U D	X Y
00.00E+00	48.372	69.688	10.000
	52.304E-25	3.0233	12000.
	-485.60	111.25	
	1.8754		
10.00	33.632	67.549	28.767
	63.876E-25	2.1020	10944.
	-456.86	101.04	
	2.2903		
20.00	28.361	66.683	37.726
	78.009E-25	1.7726	9963.5
	-434.41	95.543	
	2.7971		
30.00	27.234	66.354	42.828
	95.269E-25	1.7021	9027.6
	-412.20	91.754	
	3.4159		
40.00	28.048	66.260	46.605
	11.635E-24	1.7530	8126.9
	-387.06	88.354	
	4.1717		
50.00	29.874	66.278	50.156
	14.209E-24	1.8671	7261.1
	-356.87	84.733	
	5.0947		
60.00	32.209	66.369	53.936
	17.353E-24	2.0130	6433.8
	-319.57	80.588	
	6.2218		
70.00	34.397	66.549	57.999
	21.192E-24	2.1498	5651.2
	-272.48	75.804	
	7.5984		
80.00	34.740	66.916	61.793
	25.881E-24	2.1713	4919.1
	-211.19	70.544	
	9.2796		
90.00	28.224	67.764	63.219
	31.607E-24	1.7640	4238.7
	-127.02	65.725	
	11.333		
100.0	44.435E-03	69.955	55.718
	38.600E-24	27.772E-04	3593.2
	-75.928E-03	64.461	
	13.840		