Aggregation of Oscillating Subsystems

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ABSTRACT

In previous work, the analysis of the effects of aggregating simple dynamic systems has been studied by applying methods developed for thermodynamic systems in order to take account of stochastic effects. This approach is based on the Master Equation for the probability density of the contents of a vector of system levels. The goal of these studies is to determine the dynamic characteristics of systems composed of a population of sub-systems with the same dynamic structure while accounting for novel behavior that is introduced by the process of aggregating the sub-systems into the larger system.

In this paper, the Master Equation analysis is applied to four versions of a Commodity Cycle model to determine the nature of modes of behavior that arise from the process of aggregating a population of entities whose dynamic structure is derived from the oscillatory structure of the commodity cycle model. The approach used here is novel in two respects. It contrasts with the more recently developed analysis of chaotic systems in which non-linear, aggregate or lumped - parameter models generate behavior that is unpredictable while not being stochastic. In those models, no attempt is made to explain the large-scale or aggregate chaotic behavior in terms of the sub-systems. Compared to previous work in the same vein, this paper addresses itself to a slightly larger model as part of a natural progression in the analysis of ever-more complex systems by Master Equation methods.

INTRODUCTION

One of the fundamental problems underlying the analysis of complex systems is the relationship between the structure and behavior of the aggregate system and the structure and behavior of the sub-systems which comprise the larger system. In previous papers on this theme (Rahn 1985, Rahn 1983), the author has proposed an approach based on an interpretation of the system dynamic equations as representing the lowest order term in an expansion of the Master Equation for the joint probability distribution of the levels comprising the system state. The result of the expansion is a consistent treatment of the equations for the most-probable evolution path of the system and for the mean and variance of fluctuations about this path (van Kampen 1961, Tomita et al 1974, Portnow et al 1976).

The present paper applies the same methods to four small models of the behavior of commodity cycles (Meadows 1970, Goodman 1974). These models all include an oscillatory structure but differ in their treatment of non-linear structure, number of levels and nature of other loops (minor and major). The analysis of these models completes another step in the research program proposed in (Rahn 1985).

As in the previously mentioned analyses, some general characteristics of the method are demonstrated. For example, to lowest order in the expansion scheme used, the fluctuations about the most probable path behave similarly to the macroscopic system. That is, in linear systems the eigenvalues of the fluctuation equations are the same as for the most-probable path; in non-linear systems, the variance of the fluctuations grows during periods of expansion or local instability and stabilizes when the macrocsopic system reaches equilibrium. More detailed evaluation of the characteristics of the fluctuations is reserved for the discussion of each case.

MODEL I

The first model is composed of two levels representing mature and juvenile stocks of animals linked by a maturation delay. The birth rate of juveniles is an increasing function of the market price for mature animals and is proportional to the number of mature animals. The consumption rate of mature animals is a decreasing function of the market price and proportional to the (constant) population. The equations of the model in differential form are:

$$\left.\begin{array}{ll} \dot{Y} = -mY + FBRN*FBRM(P)*A & \text{JUVENILES} \\ \dot{A} = mY - POP*PCC(P) & \text{ADULTS} \\ P(A) = p_0 - p_1 A & \text{MARKET PRICE} \\ FBRM(P) = b_0 + b_1 P & \text{FRACTIONAL BIRTH RATE MULTIPLIER} \\ PCC(P) = c_0 - c_1 P & \text{PER CAPITA CONSUMPTION} \end{array}\right\} \quad (1)$$

In this form, non-linear table functions for P, FBRM and PCC are linearized about the current value of the level vector. The factor m in the dynamic equations is the reciprocal of the maturation delay time. Substituting single symbols for complicated combinations of constants, we can write the dynamic equations more simply as:

$$\left.\begin{array}{l} \dot{Y} = -mY + y_1 A - y_2 A^2 \\ \dot{A} = mY - a_1 A - a_0 \end{array}\right\} \quad (2)$$

To interpret these equations as defining a stochastic birth and death process, we define a set of elementary tragsitions or changes in the number of Y and A elements and the associated probabilities of such transitions. In this model, it is natural to specify that the transition that removes one unit of Y (by maturation) simultaneously adds one unit to A; i.e. the transitions are not independent of each other. With this specification, we have the transition table shown below where $\underline{x}$ is the vector of levels (Y,A) and $\delta\underline{x}$ is the transition vector.

Transition	Probability $w(\underline{x},\delta\underline{x})$
$\delta Y = +1, \delta A = 0$	$y_1 A - y_2 A^2$
$\delta Y = 0, \delta A = +1$	$a_1 A + a_0$
$\delta Y = -1, \delta A = +1$	mY

From these specifications, we derive the macrosopic rate of change for the most-probable path (maximum of the distribution of the state vector, $\underline{y}$):

$$c_1(\underline{y}) = \begin{bmatrix} \sum_{\delta A} \sum_{\delta Y} \delta Y w(\underline{y},\delta\underline{x}) \\ \sum_{\delta A} \sum_{\delta Y} \delta A w(\underline{y},\delta\underline{x}) \end{bmatrix} \begin{bmatrix} -mY + y_1 A - y_2 A^2 \\ mY - a_1 A - a_0 \end{bmatrix} \quad (3)$$

We see that the macroscopic equation reproduces the original system equations as expected. From equation 3, we can derive the system matrix for the mean of the fluctuations, $\underline{\mu}$:

$$K_{ij} = \frac{\partial c_1^i(\underline{y})}{\partial y_j} = \begin{bmatrix} -m & y_1 - 2y_2 A \\ m & -a_1 \end{bmatrix} \quad (4)$$

The system matrix for the variance-covariance matrix of the fluctuations is found from the equation:

$$\frac{d\sigma}{dt} = K\sigma + (K\sigma)^T + D \quad (5)$$

where D is the matrix of second moments of the transition probabilities and is a function of $\underline{y}$, the state vector of the most-probable path. The characteristics of the mean and covariances are summarized by the eigenvalues of the corresponding system matrices. For the mean, we have the characteristic equation:

$$\lambda^2+(a_1+m)\lambda+m(a_1-(y_1-2y_2A))=0 \quad (6)$$

from which we see that there are oscillations iff

$$(a_1+m)^2<4m(a_1-(y_1-2y_2A)) \text{ or } (a_1-m)^2<4m(2y_2A-y_1) \quad (7)$$

and there is an unstable root if

$$-4m(a_1 - (y_1 - 2y_2A)) > 0$$

or equivalently if

$$0 < A < \frac{y_1 - a_1}{2y_2}$$

The numerator of the right-hand inequality can be re-written in terms of the orginal model variables as:

$$FBRN.(b_0 + b_1.p_0) > POP.c_1.p_1$$

This latter relationship states the reasonable condition that unstable growth modes can occur if the birth rate is greater than the consumption rate while the limit on the size of the mature population means that the instability will be limited by the growth of the mature population level.

For the covariances, the eigenvalue equation is

$$(\lambda + a_1 + m + \lambda)[(\lambda + 2m)(\lambda + 2a_1) - 4md_2] = 0$$

whose solutions are:

$$\lambda = -(a_1 + m); \ \lambda = -(a_1+m) \pm [(a_1+m)^2 + 4md_2]^{1/2}$$

where $d_2 = y_1 - 2y_2A$

Comparison of this result with equation 7 shows that the condition for oscillations to occur is less restrictive for the mean than for the covariances of the fluctuations. We see also that the covariances are unstable if

$$0 < 4md_2 = 4m(y_1 - 2y_2A) < 4m(y_1 - y_2A)$$

i.e., instability may occur only for small values of the mature population, at which time the system is in a growth phase.

MODEL II

The second model is a simplified version of a commodity cycle model in (Goodman 1974); the main simplification consists in suppressing the use of an expected market price in favor of the market price itself. This modification eliminates a delay and thus a minor negative loop. The dynamic equations are:

$\dot{PC} = (DPC - PC)/CAD$	PRODUCTION CAPACITY	
$\dot{DI} = PR - CR$	DISTRIBUTOR'S INVENTORY	
$DPC = d_0 + d_1P$	DESIRED PRODUCTION CAPACITY	(8)
$PR = PC$	PRODUCTION RATE	
$CR = POP.(c_0 - c_1P)$	CONSUMPTION RATE	
$P = p_0 - p_1DI$	MARKET PRICE	

After substitution we get the system equations for PC, DI:

$$\dot{PC} = (d_0+d_1(p_0-p_1DI)-PC)/CAD \qquad (9)$$
$$\dot{DI} = PC - POP.(c_0-c_1(p_0-p_1DI))$$

In this model the transitions of PC and DI are mutually independent. One other feature of interest is the fact that the single term in the rate of change of PC can be positive or negative so that allowance must be made for this possibility in the specification of the transition probabilities. It will be seen that this feature has no effect on the characteristics of the dynamics of the mean and covariance of the fluctuations.

Transition	Probability $w(\underline{x},\delta\underline{x})$
$\delta PC = +1, \delta DI = 0$	$p(PC<DPC)(d_0+d_1(p_0-p_1DI)-PC)/CAD$
$\delta PC = -1, \delta DI = 0$	$p(PC>DPC)(PC-(d_0+d_1(p_0-p_1DI))/CAD$
$\delta PC = 0, \quad \delta DI = +1$	PC
$\delta PC = 0, \quad \delta DI = -1$	$POP.(c_0-c_1(p_0-p_1DI))$

The vector of the macroscopic rate of change reproduces the linearized system equations since the sum of the probabilities:

$$p(PC<DPC) + p(PC>DPC) = 1.$$

The system matrix for the mean of the fluctuations is:

$$K_{ij} = \begin{pmatrix} -1/CAD & -d_1p_1/CAD \\ 1 & -POPc_1p_1 \end{pmatrix}$$

From this matrix we can determine that the fluctuations are always stable and if there are oscillations, they are at the same frequency as the macroscopic system.

For the variance-covariances of the fluctuations, the eigenvalue equation is a cubic:

$$(\lambda+POPc_1p_1+1/CAD)\left[(\lambda+2/CAD)(\lambda+2.POPc_1p_1)+4d_1p_1/CAD\right]=0 \qquad (10)$$

We note that this equation factors into a real, negative root and a quadratic term as in Model I and as in the Workforce-Inventory model in (Rahn 1983) in spite of differences in both the dynamic and the stochastic models. The robustness of this feature is due to the linearity of the underlying models. By substituting in equation 10:

$$a = 1/CAD$$
$$b = POPc_1p_1$$
$$D = d_1p_1/CAD$$

we can re-write it as

$$(\lambda+a+b)[(\lambda+2a)(\lambda+2b)+4D] = 0 \qquad (11)$$

whose solutions are:

$$\lambda_1 = -(a+b)$$

$$\lambda_{\pm} = -(a+b) \pm [(a+b)^2 - 4(ab+D)]^{1/2} = -(a+b) \pm [(a-b)^2 - 4D]^{1/2}$$

The mean and covariance components decay exponentially with or without oscillations under the same conditions as the macroscopic, linear model. This result was first derived for the case of a linear Workforce-Inventory model in (Rahn 1983).

MODEL III

The third model is a simplification of the generic commodity cycle model presented in (Meadows 1970). The main simplification consists in eliminating the expected consumption rate function (a delay) in favor of the consumption rate as a function of market price. The main difference with respect to the previous model is the inclusion of the expected market price as an explicit level so that the model has three levels. The dynamic equations in differential form are:

$\dot{PC} = (DPC - PC)/CAD - PC/ALPC$	PRODUCTION CAPACITY	
$\dot{DI} = PR - CR$	DISTRIBUTOR INVENTORY	
$\dot{EP} = (P - EP)/EPAD$	EXPECTED MARKET PRICE	
$P = P(DI) = p_0 - p_1 DI$	MARKET PRICE	
$DPC = d_0 + d_1 EP$	DESIRED PRODUCTION CAPACITY	(12)
$PR = PC$	PRODUCTION RATE	
$CR = POP*PCC(P)$	CONSUMPTION RATE	
$PCC = c_0 - c_1 P$	PER CAPITA CONSUMPTION RATE	

After substitution of the linearized forms of the table functions in equation 12 we get the system equations:

$$\dot{PC} = (d_0 + d_1 EP - PC)/CAD - PC/ALPC$$

$$\dot{DI} = PC - POP*(c_0 - c_1(p_0 - p_1 DI))$$

$$\dot{EP} = ((p_0 - p_1 DI) - EP)/EPAD$$

The transitions are again considered to be mutually independent and we propose the following stochastic model:

Transition	Probability $w(\underline{x},\delta\underline{x})$
$\delta PC = +1, \delta DI = 0, \delta EP = 0$	$p(PC<DPC)(d_0+d_1EP-PC)/CAD$
$\delta PC = -1, \delta DI = 0, \delta EP = 0$	$p(PC>DPC)(PC-d_0-d_1EP)/CAD+PC/ALPC$
$\delta PC = 0, \delta DI = +1, \delta EP = 0$	PC
$\delta PC = 0, \delta DI = -1, \delta EP = 0$	$POP.(c_0-c_1(p_0-p_1DI))$
$\delta PC = 0, \delta DI = 0, \delta EP = 1$	$p(EP<P)(p_0-p_1DI-EP)/EPAD$
$\delta PC = 0, \delta DI = 0, \delta EP = -1$	$p(EP>P)(EP-p_0+p_1DI)/EPAD$

The macroscopic rate of change, $c_1(\underline{y})$, reproduces the linearized system equations for the same reason as in Model II. The system matrix for the mean of the fluctuations is:

$$K_{ij} = \begin{bmatrix} -1/CAD - 1/ALPC & 0 & d_1/CAD \\ 1 & -POPc_1p_1 & 0 \\ 0 & -p_1/ECAD & -1/ECAD \end{bmatrix}$$

The eigenvalue equation for the mean fluctuation, μ, is

$$(\lambda + 1/CAD + 1/ALPC)(\lambda + POPc_1p_1)(\lambda + 1/EPAD) - D = 0$$

The eigenvalue equation for the variance-covariances is a very complicated sixth order polynomial in factors of $(\lambda+...)$ whose constant term is $-8D^2$. For both the mean and variances, the constant term acts to perturb the negative eigenvalue of smallest absolute value in the direction of less stability, towards smaller absolute values or even to destabilize it. The constant term, by shifting the eigenvalue function vertically downward, may cause some real roots to coalesce in complex conjugate pairs which are stable.

MODEL IV

The final model is another simplification of the generic commodity cycle model in (Meadows 1970). In this version, the expected market price is replaced by the market price and the expected consumption rate is included in order to have an explicit non-linearity in the dynamic equations:

$\dot{PC} = (DPC-PC)/CAD - PC/ALPC$	PRODUCTION CAPACITY
$\dot{DI} = PC - POP*PCC(P)$	DISTRIBUTOR INVENTORY
$\dot{ECR} = (CR-ECR)/ECAD$	EXPECTED CONSUMPTION RATE
$P = P(COV/DCOV) = \hat{p}_0-\hat{p}_1DI/ECR$	MARKET PRICE
$PCC = c_0-c_1(p_0-\hat{p}_1DI/ECR)$	PER CAPITA CONSUMPTION RATE
$DPC = d_0+d_1P$	DESIRED PRODUCTION CAPACITY

The system matrix for the mean of the fluctuations is

$$K_{ij} = \begin{bmatrix} -1/CAD-1/ALPC & -D_1 & D_1 DI/ECR \\ 1 & -P_1 & P_1 \dot{D}I/ECR \\ 0 & P_1/ECAD & -(P_1 DI/ECR + 1)/ECAD \end{bmatrix}$$

where

$$D_1 = d_1 \hat{p}_1/(ECR.CAD); \quad P_1 = POPc_1 \hat{p}_1/ECR$$

The stability of the fluctuations is determined by a complicated cubic whose constant term is 'destabilizing' in the sense mentioned previously for Model III; i.e. the real, negative root of smallest absolute value is displaced to the right by the downward shift imposed by the constant term.

CONCLUSIONS

In all of these commodity cycle models except for Model II, there is a possibility for the mean and/or the variance-covariances of the fluctuations to become unstable. Such a development should make itself evident as a modification of the macroscopic behavior mode unless other factors such as non-linearity disrupt the behavior expected on the basis of the foregoing analysis. The modest range of models that has been presented in this series of papers indicates that there are some situations in which fluctuations may grow exponentially. It remains to verify these possibilities by simulation.

Models of third order for the mean of the fluctuations about the most-probable path seem to be at the limit of intuitive analysis especially as regards the evolution of the variance-covariances which give a sixth order polynomial for the eigenvalue equation. Further effort to analyze more realistic models will require more powerful tools.

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