GENERIC QUALITATIVE BEHAVIOR OF ELEMENTARY SYSTEM DYNAMICS STRUCTURES

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Abstract

The paper starts with some reflections on the qualitative nature of the concepts involved in system dynamics modelling. To deal rigorously with this nature *topological* tools are needed. These tools are being developed around the *qualitative theory of nonlinear dynamical systems*. The relevance of this theory to system dynamics is now begining to be understood.

The paper introduces a dynamical system associated to the causal diagram that contains only qualitative information. Some interesting results on the qualitative behavior of the system can be obtained from this dynamical system.

1 Introduction

In the system dynamics context the word *qualitative* can be used with at least two senses:

- In the first one qualitative is synonymous to pre-quantitative or poorly quantitative. In this sense it is said that the causal or influence diagram contains only qualitative information. The qualitative analysis of a system dynamics model can consist of the elucidation of the feedback loops, the determination of the sign of these loops, and of the character of self-regulation or of explosive behavior associated with them.
- In the other one, the word qualitative is used in a very formal and rigorous way, the one proposed by the *qualitative theory of dynamical systems* (Guckenheimer and Holmes 1983). This use has deep topological and geometrical connotations (Abraham and Shaw 1987) and is based on the concept of qualitative that has its roots in the work of Poincaré, that has been updated by Thom (Thom 1977, 4-7) and Zeeman (Zeeman 1977, 319-329).

Wosthelholme has proposed calling the first use *qualitative system dynamics* (Wosthelholme 1985). However, this denomination is restrictive if used only in this sense. It should include both meanings, as we propose in this paper.

In so far as a system dynamics model is a mathematical object known as a *dynamical system*, the results of the qualitative theory of dynamical systems can be applied straightforwardly to them. This has been done in (Aracil 1981, 1984, 1986; Mosekilde, Aracil and Allen 1988), where the relevance of these results for the system dynamics method has been emphasized. In these references it is suggested that a *generalised sensitivity analysis* of system dynamics models can be developed. This qualitative analysis allows a better understanding of how the behavior of the system is produced, and of how to control this behavior.

The qualitative analysis is based mainly on tools of a geometrical nature. These tools take advantage of the graphical possibilities of computers. They help to develop a deep intuitive comprehension of the mechanism underlying the behavior of the model, and, in this way, to understand how the behavior is generated, and how to act in order to modify it.

The actual implementation of qualitative analysis techniques is practically restricted to small size models. However this is not a great disadvantage as in recent times greater emphasis has been placed on small models in the system dynamics community (Forrester 1986, Morecroft 1988). These small models show a greater transparency for the dialogue between experts 'mental model' and 'simulation model behavior'. This dialogue can be greatly extended with the tools supplied by the qualitative analysis techniques.

In this paper we will assume that the qualitative information about a given concrete system comprises no numerical information beyond the signs of the influences, the relative value of these influences and the classification of the variables in a system as levels, rates and auxiliaries. With this information we try to get as much knowledge as possible on the behavior modes of the system, in the concrete meaning given to behavior mode in the qualitative theory of nonlinear dynamical systems.

Our aim is to explore how the formal qualitative analysis techniques, based on the second of the above senses of qualitative, can be used to solve the kind of questions suggested by the first of the uses. In this way a synthesis of both senses can be reached.

The results here reported are still in a work progress stage. For instance, a computer inplementation of them is being developed. However we think that they are interesting enough to deserve publication.

2 A simple dynamical system associated to a causal diagram

Assume we have got the causal diagram of a model, and that we have classified all the variables appearing in it as levels, rates and auxiliaries. This latter classification involves a knowledge of the structure of the system that is rather more involved than the one in the causal diagram. However, it is still of a qualitative (in the classical sense of this word) nature: it still does not involve any quantitative knowledge. The aim of this paper is to state what can be said about the behavior of the system from this knowledge.

We can associate a signed, directed graph (SDG) to a causal diagram. In this graph closed loops can be found. Classical qualitative analysis (in the Wosthelholme sense) is mainly concerned with this search, and goes on to elucidate the relationship between these loops and the main characteristics of the behavior.

Once the variables of the causal diagram have been classified into levels, rates and auxiliares we know that the mathematical form of the model takes the form:

$$\begin{aligned} \dot{x} &= Ar \\ r &= f_r(x, z, p) \\ z &= f_z(x, p) \end{aligned} \tag{1}$$

Where $x \in R^n$ stands for the level or state variables, $r \in R^m$ for the rate variables, $z \in R^s$ for the auxiliary variables and $p \in R^q$ for the parameters. Matrix A is $n \times m$, where n and m are the number of state and rate variables respectively, with $a_{ij} = 1$ if r_j influences positively on x_i, $a_{ij} = -1$ if it influences negatively, and $a_{ij} = 0$ it there is no influence.

The functions f_r (resp. f_z) give the value of a rate variable r_i (resp. z_i) from the value of the state variables x, the auxiliary variables z and the parameters p that influence r_i (resp. z_i). In this preliminary stage of the modelling process the concrete mathematical form of these functions will not be known. If the modelling process has only reached the causal diagram stage we only know if the influence exists and which is the sign of the influence. At this stage of the modelling process we know that every variable u (rate or auxiliary) depends upon some other variables (and eventually parameters). This means that there is a multivariate causality acting upon u, as shown in Fig. 1.

According to conventional system dynamics this multivariate causality can be given a separable multiplicative formulation. Then, it can be written:

$$\begin{aligned} u &= f(y_1, y_2, ..., y_k) \\ &= u_n \times f_1\left(\frac{y_1}{y_{1n}}\right) \times f_2\left(\frac{y_2}{y_{2n}}\right) \times ... \times f_k\left(\frac{y_k}{y_{kn}}\right) \end{aligned}$$

where the functions f_i are the well known system dynamics *multipliers* (Forrester 1969, p. 22-30), and u_n, y_{in},... y_{kn} stand for the normal values of variables u, y_i,... y_k respectively.

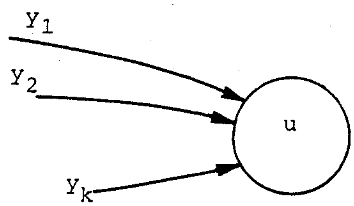

Figure 1: The variable u depends on other variables y_i

For the moment, the only assumption on functions f_i is that they are monotone. This is consistent with the fact that we can give a sign to the relation (to the arrow in the causal diagram) that relates y_i with u (Fig. 1). The general qualitative characteristics of the functions f_i are:

$$f_i(x) > 0 \ \ \forall x, \qquad f_i(\infty) < k, \qquad f_i(1) = 1,$$

$$y_i \xrightarrow{+} u \Rightarrow \frac{d}{dy} f_i\left(\frac{y_i}{y_{in}}\right) > 0 \qquad y_i \xrightarrow{-} u \Rightarrow \frac{d}{dy} f_i\left(\frac{y_i}{y_{in}}\right) < 0$$

The slope of the function $f_i(x)$ at $x = x_k$ is given the name of *intensity* of the relation. It can be considered as a measure of the strength of the influence relation for this value of x. It will be assumed that the intensity of the influences of the rates on the states is + 1 or - 1.

Consider the SDG of a causal diagram. In this graph a sequence of nodes and branches (that is, of influences) gives rise to a *path*. If we give a number to every node of the SDG, then the path linking consecutive nodes i, j and k will be denoted by (ijk). The intensity of the path is defined as the product of the intensities of the relations that form it and will be denoted by $w(ijk)$.

A remarkable property of this definition of intensity of a path is the following. The Jacobian matrix of dynamical system (1) can be written

$$J = D_x f = A[D_x f_r + (D_z f_r)(D_x f_z)]$$

where J is an $n \times n$ matrix. Let $B = A(D_x f_r)$ and $C = A(D_z f_r)(D_x f_z)$, then $J = B + C$.

It is clear that $B_{ij} = \sum_k A_{ik}(D_x f_r)_{kj}$ where A_{ik} is the intensity of the influence relation $r_k \longrightarrow x_i$ and $(D_x f_r)_{kj}$ is the intensity of the influence relation $x_j \longrightarrow r_k$. Then the element $A_{ik}(D_x f_r)_{kj}$ is the intensity of the path $x_j \longrightarrow r_k \longrightarrow x_i$. Therefore B_{ij} is the sum of the intensities of all the paths that link the variable x_i with x_j through a rate variable.

In the same way, $C_{ij} = \sum_{k,l} A_{ik}(D_z f_r)_{kl}(D_x f z)_{lj}$, where A_{ik} has the same meaning as in B; $(D_z f_r)_{kl}$ is the intensity of the influence $z_l \longrightarrow r_k$; and $(D_x f_z)_{lj}$ is the intensity of $x_j \longrightarrow z_l$. Therefore $A_{ik}(D_z f_r)_{kl}(D_x f_z)_{lk}$ is the intensity of the path $x_j \longrightarrow z_l \longrightarrow r_k \longrightarrow x_i$ and C_{ij} is the sum with respect to all k (all the rates) and all l (all the auxiliary variables). Then C_{ij} is the sum of the intensities of all the paths that link x_j with x_i through at least an auxiliary variable.

As far as $J_{ij} = B_{ij} + C_{ij}$, then J_{ij} is the sum of the intensities of all paths that link x_j with x_i (through auxiliary variables or not). Therefore, the element J_{ij} is the sum of the intensities of all the paths that start in the level variable j and end in the i, and it is zero if there is no path from i to j. According to this property the only information needed to get the Jacobian matrix of a system dynamics model is supplied by the causal diagram and a measure of the relative intensity of the relations in that diagram. This is a very remarkable property for qualitative analysis, as far as the Jacobian matrix incorporates a huge amount of information on the qualitative behavior (specially, on the stability properties) of a dynamical system. Some examples will illustrate this fact in the next Section.

Other related property based on the *implicit function theorem* (Poston and Steward 1978) is the following. If for some values of parameter $p \in \Omega$ the dynamical system (1) has a single equilibrium and the sign of det J does not change for $\forall p \in \Omega$, then the system has a single equilibrium for those values of $p \in \Omega$.

3 Examples

To illustrate the previous results we include some examples in this Section. For notacional simplicity in the figures representing the SDG all the variables will be given a number from 1 to $n+m+s$, where $x \in R^n$, $r \in R^m$ and $z \in R^s$ taking the state variables for 1 to n, rates from $n+1$ to $n+m$ and auxiliaries from $n+m+1$ to $n+m+s$. A path will be denoted by $(n_l...n_k)$, and its intensity $w(n_l...n_k)$. The influence relation from n_2 on n_1 will be written $f_{n_2 n_1}$.

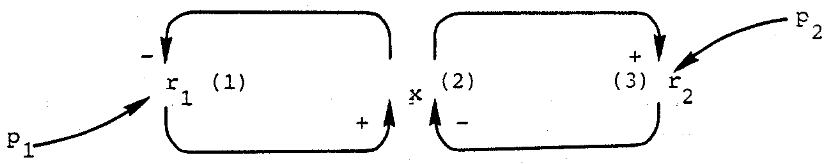

Figure 2: Causal diagram of system in Example 1

3.1 Example 1

Consider the causal diagram of Fig. 2. It represents a state (level) variable with two rates r_1 and r_2, each one affected by a parameter p_1 and p_2, respectively. The corresponding equations are:

$$\begin{aligned} r_1 &= r_{1n} f_{21}(x/x_n) \\ r_2 &= r_{2n} f_{31}(x/x_n) \\ \dot{x} &= r_1 - r_2 = f(x,p) \end{aligned}$$

If $x_n = r_{1n} = r_{2n} = 1$ then the equilibria of the system are given by the solutions to the equation

$$f_{21}(x) - f_{31}(x) = 0$$

According to the monotonicity property of the multipliers there is the single solution $x = 1$ to this equation.

The only element of the 1×1 Jacobian matrix of the system is:

$$J = w(121) + w(131) < 0$$

As far as $J < 0$ for all x then one concludes that:

- there is a single equilibrium for all p.
- this equilibrium is stable.

This is all that can be said about the qualitative nature of the long-time behavior of the model.

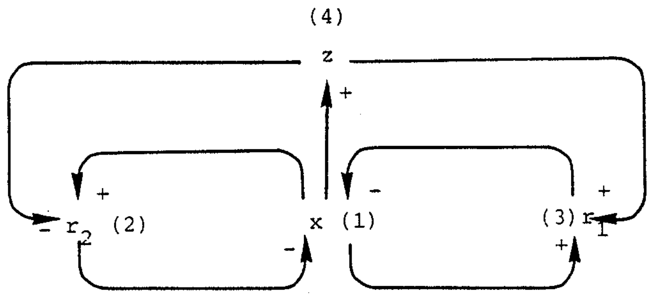

Figure 3: Causal diagram of system in Example 2

3.2 Example 2

Consider now the causal diagram of Fig. 3. That diagram is the same as in Fig. 2, with the addition of auxiliary variable z. The equations are:

$$\begin{aligned} z &= z_n f_{41}(x/x_n) \\ r_1 &= r_{1n} f_{31}(x/x_n) f_{34}(z/z_n) \\ r_2 &= r_{2n} f_{21}(x/x_n) f_{24}(z/z_n) \\ \dot{x} &= r_1 - r_2 = f(x,p) \end{aligned}$$

As in Example 1, if the normal values are $x_n = z_n = r_{1n} = r_{2n} = 1$, then the equilibria of the system are the solutions to the equation

$$f(x) = f_{21}(x) f_{24}(f_{41}(x)) - f_{31}(x) f_{34}(f_{41}(x)) = 0$$

This equation has the solution $x = 1$. This is a single solution due to the properties of functions f_{ij}.

The Jacobian is

$$J = D_x f = w(121) + w(1421) + w(131) + w(1431) < 0$$

Then there is a single and stable equilibrium for all values of the parameters.

3.3 Example 3

The last case to be considered is the one in Fig. 4. The corresponding equations are:

$$\begin{aligned}
r_1 &= r_{1n} f_{32}(x_2/x_{2n}) f_{31}(x_1/x_{1n}) \\
r_2 &= r_{2n} f_{42}(x_2/x_{2n}) f_{41}(x_1/x_{1n}) \\
r_3 &= r_{3n} \\
r_4 &= r_{4n} f_{61}(x_1/x_{1n}) \\
\dot{x}_1 &= r_1 - r_2 \\
\dot{x}_2 &= r_3 - r_4
\end{aligned}$$

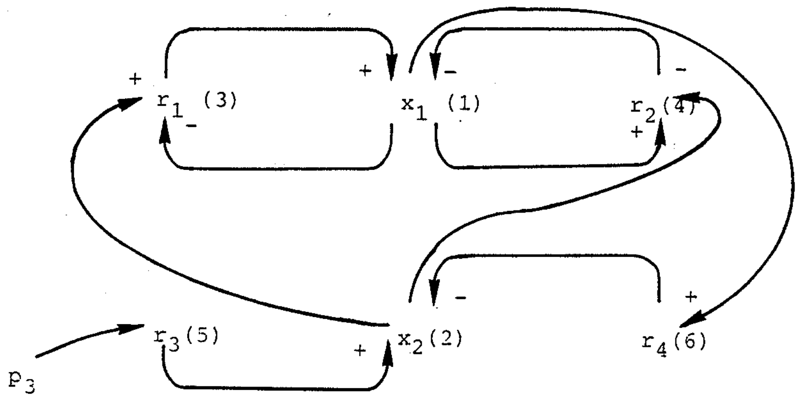

Figure 4: Causal diagram of system in Example 3

For the normal values $x_{1n} = x_{2n} = r_{1n} = r_{2n} = r_{3n} = r_{4n} = 1$, the equilibria of the system are given by the equations:

$$\begin{aligned}
f_{32}(x_2) f_{31}(x_1) - f_{42}(x_2) f_{41}(x_1) &= 0 \\
1 - f_{61}(x_1) &= 0
\end{aligned}$$

The only solution to this system is $x_1 = x_2 = 1$.

The Jacobian matrix of the system is:

$$J = \begin{pmatrix} w(131) + w(141) & w(231) + w(241) \\ w(162) & 0 \end{pmatrix}$$

The characteristic polynomial is $\phi_J = \lambda^2 + a_1\lambda + a_2$, with $a_1 = -J_{11}$ and $a_2 = J_{12}J_{21}$. It is easy to see that $a_1 > 0$, $a_2 > 0$, $\forall x_i$. Therefore there is a single equilibrium for all values of the parameters, which, furthermore, is stable.

Supose now that the relation $x_1 \longrightarrow r_1$ changes from negative to positive. Then, again, there is a single equilibrium, as $a_2 > 0$; but, now $a_1 = w(131) + w(141)$ has no definite sign. If $| w(131) | < | w(141) |$ then $a_1 < 0$ and the equilibrium is stable. This means that if loop (141) dominates over loop (131) the equilibrium is stable. As $| w(141) |$ decreases and $| w(131) |$ increases, a_1 decreases until it becomes negative. Then the equilibrium becomes unstable, and a Hopf bifurcation is produced (in the generic case) giving rise to a stable limit cycle. The long time behavior has commuted from steady to oscillatory.

4 Conclusions

The qualitative nature of the knowledge considered and of the conclusions reached in the examples in last section should be noticed.

The procedure proposed in this paper should lead to a test on the causal diagram (once the variables have been classified in levels, rates and auxiliaries) to state the different behavior modes the modeller should expect from the model. A computer inplementation of the test is in progress.

5 References

R.H. Abraham and C. D. Shaw, 1987, "Dynamics: a visual introduction", in *Self-Organizing Systems*, F.E. Yates (Ed.), 541-597 Plenum Press.

J. Aracil, 1981, "Structural Stability of low-order System Dynamics models", *Int. J. System Science*, vol. 12, 423-441.

J.Aracil, 1984, "Qualitative Analysis and Bifurcations in System Dynamics Models", *IEEE-SMC-14*, No. 4, 688-696.

J. Aracil, 1986, "Bifurcations and Structural Stability in the Dynamical Systems Modeling Process"; *Systems Research*, vol. 3, No. 4, 243-252.

J.W. Forrester, 1969, *Urban Dynamics*, Wright-Allen Press.

J.W. Forrester, 1986, "Lessons from System Dynamics modeling", *1986 System Dynamics Conference*, Sevilla, October 22-24, 1-16.

J. Guckenheimer and P. Holmes, 1983, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer-Verlag.

Morecroft, 1988, "System Dynamics and microworlds for policy makers", *European Journal of Operational Research*, 35, 301-320.

Mosekilde,E., J. Aracil and P. Allen. 1988., "Instabilities and chaos in non-linear dynamic systems", *System Dynamics Review*, Vol. 4, 14-55.

T. Poston and I.N. Steward, 1978, *Catastrophe Theory and its Applications*, Pitman.

Rene Thom, 1977, *Stabilité structurelle et morphogénese*, Inter Editions, París.

E.F. Wolstenholme, 1985, "A methodology for Qualitative System Dynamics", *1985 System Dynamics Conference*, pp. 1049-1057.

E.C. Zeeman, 1977, *Catastrophe theory: selected papers*, Addison-Wesley.