RÖSSLER BANDS IN ECONOMIC AND BIOLOGICAL SYSTEMS

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Abstract

A Rössler band is presumably the simplest of all chaotic attractors. It can develop in systems with only three state variables if two of these produce an outward spiralling trajectory, and the third folds this trajectory back towards its center when the amplitude of the expanding oscillation becomes sufficiently large. In the present paper we show how Rössler bands can develop through slight modifications of well-established economic and biological models.

1. Introduction

Non-linear dynamic phenomena are increasingly recognized as essential for the function of normal biological systems. Besides the beating of the heart and the ovarian cycle, the classical examples of self-sustained physiological oscillations, investigations performed during the last decade have revealed the existence of a great variety of rhythmic phenomena with periods ranging from fractions of a second to several hours or even days. Examples are hormonal regulation and neuronal function, with secretion of insulin (1), growth hormone, and luteinizing hormone (2) all showing periods of 2-3 hours. A somewhat similar period is observed for the production of enzymes in certain bacteria (3).

In the macroeconomic realm, the economic long wave as depicted in the Sterman model (4) is a typical example of a self-sustained oscillation. It is also clear that other modes of the macroeconomic system can turn unstable and produce highly non-linear dynamic phenomena. The ordinary 3-7 year business cycle thus appears to become destablized during the later stages of a long-wave upswing (4).

We have worked with this type of phenomena for a couple of years. In particular, we have developed a model for the regulation of pressures and flows in the nephrons of the kidney (5). For normal parameter values, this model reproduces experimentally observed self-sustained oscillations in the nephron pressure. However, if the delay of the main negative feedback loop is increased, deterministic chaos develops through a Feigenbaum cascade of period-doubling bifurcations (6). This corresponds to observations for hypertensive rats and for rats which have received a light dose of furosemide, a drug which is known to influence the function of the kidney. A similar cascade of period-doubling bifurcations has been found in a model of resource allocation in a managerial system (7), and we have also shown how the economic long-wave model can become chaotic when perturbed by a sinusoidal variation in the demand for capital to the goods sector (8).

Both for the kidney model and for the managerial resource allocation model, the observed chaotic attractor resembles a Rössler band (9). This is presumably the simplest type of a strange attractor that one can think of, and the purpose of the present paper is to show how similar attractors can develop in other economic and biological models. The examples that we shall consider are slightly modified versions of the commodity market model developed by Meadows (10) and of a model of a classical microbiological control system, the tryptophan operon (3). Our aim is primarily to clarify the type of structure which can lead to Rössler-like attractors.

In its original version, the Rössler model consists of the following set of three coupled non-linear differential equations (8):

$$\frac{dx}{dt} = -y - z \quad (1a) \qquad \frac{dy}{dt} = x + ay \quad (1b) \qquad \frac{dz}{dt} = b + xz - cz \quad (1c)$$

where a, b and c are parameters. Often, a and b are kept constant at $a = b = 0.2$ while c is varied between 2 and 10. A typical Rössler band exists, for instance, for $c = 5.7$. As illustrated in figure 1, this band may be considered to be generated by a trajectory which for small z spirals outwards immediately over the xy-plane. As the amplitude of this expanding oscillation becomes sufficiently large, the term xz in the equation for dz/dt causes z to increase, and the trajectory then moves away from the xy-plane. The increase in z gives rise to a reduction of the amplitude of oscillation such that the trajectory is folded back towards the z-axis. The reduction in |x| in turn causes z to relax towards its initial small value whereafter a new outward spiralling movement is begun.

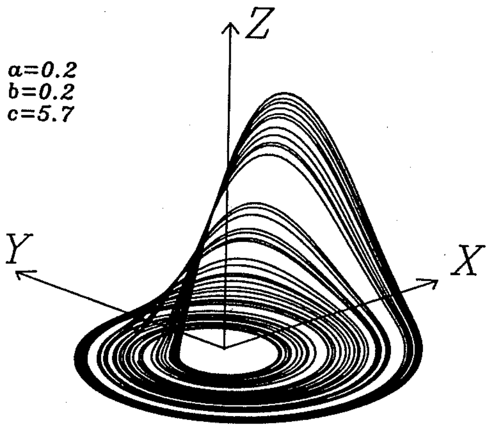

Figure 1. Rössler band obtained from Eqs. 1a-c for a = b = 0.2 and c = 5.7. The chaotic attractor arises through a Feigenbaum cascade of period-doubling bifurcations as c is increased.

2. The Commodity Market Model

Figure 2 shows the flow diagram for a slightly modified version of Meadows' commodity market model (10). Let us for a moment consider the production capacity PC to be constant. The model then contains two negative feedback loops which regulate production and consumption, respectively. In accordance with classical cobweb theory, the stability of the model is controlled by the elasticities of the demand and supply curves in the equilibrium point. These curves are represented by the table functions for indicated per capita consumption (IPCC) and for desired production (DP). A high demand elasticity and a low supply elasticity produce stable behavior. It is quite likely, however, that a commodity market in certain periods can operate under unstable conditions. Our first modification to the original model has therefore been to reduce the demand elasticity, until growing oscillations were obtained.

To restrain these expanding oscillations we have introduced a second modification: it is assumed that depreciation of production capacity depends upon capacity utilization. Thus, if the capacity utilization factor CUF averaged over the capacity adjustment time CAT is unacceptably low, the capacity depreciation rate CDR is increased by a factor DMF which typically takes a value of 2 to 6. The associated reduction in production capacity forces the model into a region

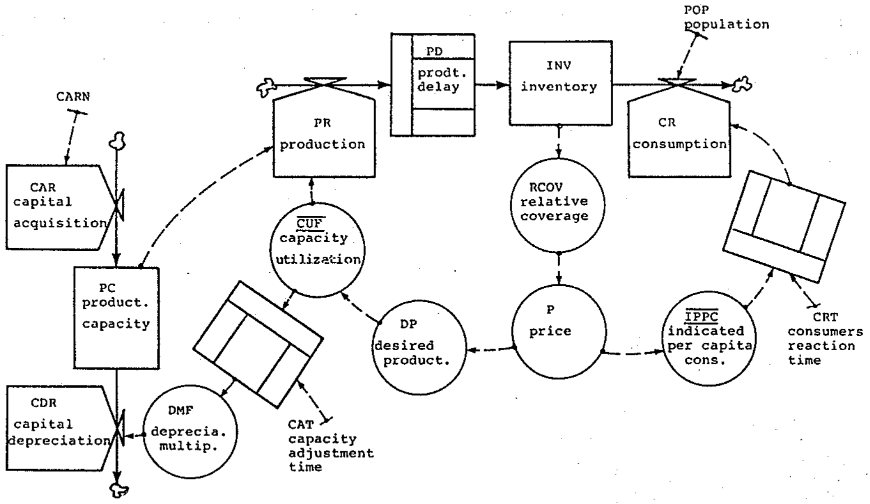

Figure 2. Slightly modified version of the classical commodity market model.

where capacity utilizaton is nearly complete and where the effective supply elasticity therefore is small. However, as the production capacity is gradually rebuilt, a new growing spiral in production and consumption is started.

Figures 3-6 present a series of simulation results obtained with the above model as the capacity adjustment time CAT gradually is increased from 1 to 5 months. Each figure shows a projection of the phase space trajectory into the plane expanded by inventory and production capacity, together with a plot of simultaneous values for

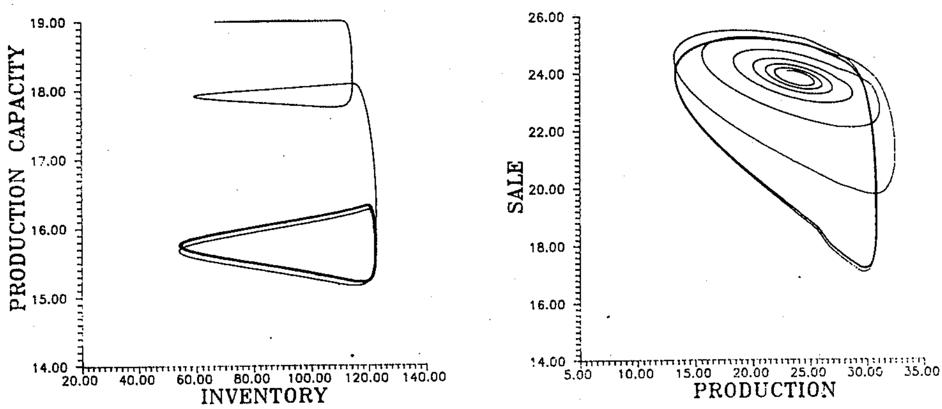

Figure 3. Transient approach to the limit cycle existing for CAT= 1 month and DMF = 4.

production and sale. Figure 3 includes the transient behavior as the model is initiated with a relatively high production capacity. Note how production and sale during this transient spiral outwards towards the stable limit cycle attractor. In the subsequent figures the transient has been left out, retaining only the stationary long-term behavior.

Figures 4 and 5 show the stable period-2 and period-4 cycles existing for CAT = 1.75 and 3.5 months, respectively. In both cases DMF = 4. It is interesting to note in figure 4 how the production capacity builds up during two oscillations of the inventory cycle until suddenly the capacity utilization becomes too low, and the production capacity is reduced by about 10% within a relatively short period of time. Figure 5 shows a somewhat similar picture except that four inventory oscillations must be completed before the model reproduces itself. As CAT is further increased, the model exhibits the usual cascade of period-doubling bifurcations to reach a chaotic regime for CAT ≅ 4.2 months. Figure 6 shows a characteristic example of a chaotic attractor. In this mode, the model is sensitive to the initial conditions. It is therefore impossible to make predictions over longer time horizons even though the equations of motion are completely deterministic.

A more complete overview of the behavior of the model is provided by the bifurcation diagram in figure 7. This diagram was obtained for DMF = 4.25 by slowly scanning CAT from 1 to 8.5 months while plotting for each oscillation of the model the maximum production attained. For low values of CAT we observe the characteristic Feigenbaum cascade of period-doubling bifurcations (6) with a transition to chaos at CAT ≅ 3 months. In the chaotic regime we note the periodic windows, particularly the very pronounced period-3 window existing for CAT > 5.7 months.

To complete our investigation of the commodity market model, figure 8 shows the distribution of stationary solutions in the parameter plane expanded by CAT and DMF. Besides the already mentioned periodic solutions this figure shows the positions in parameter plane where one can find stable period-8, period-16, period-6, period-12, and period-5 solutions. These findings are very similar to those previously reported for the managerial resource allocation model (7).

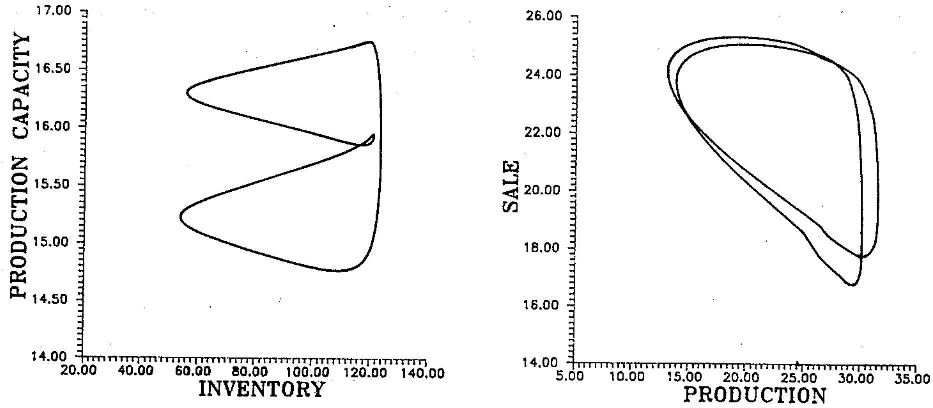

Figure 4. Stable period-2 cycle observed for CAT = 1.75 months.

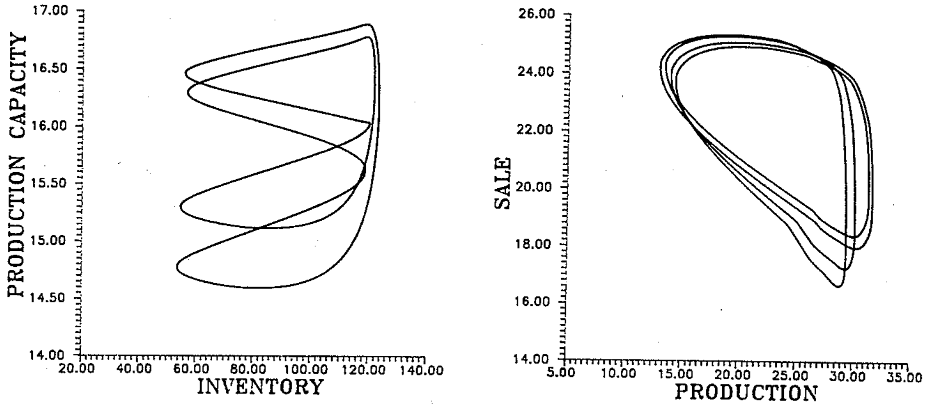

Figure 5. Stable period-4 cycle observed for CAT = 3.5 months.

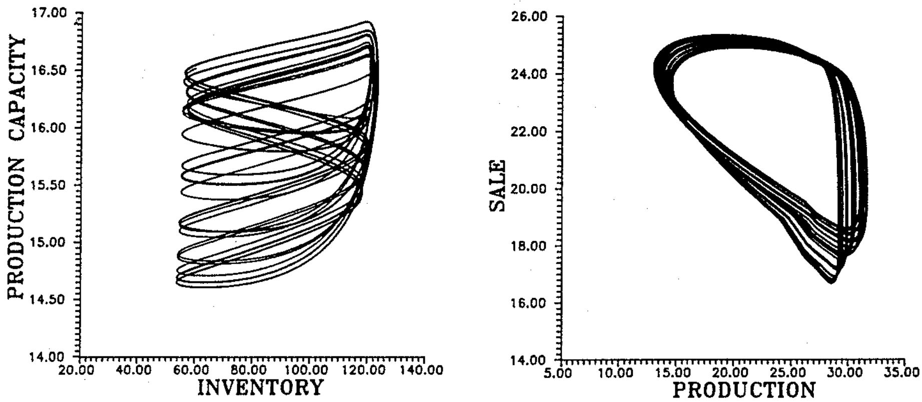

Figure 6. Chaotic attractor observed for CAT = 5 months.

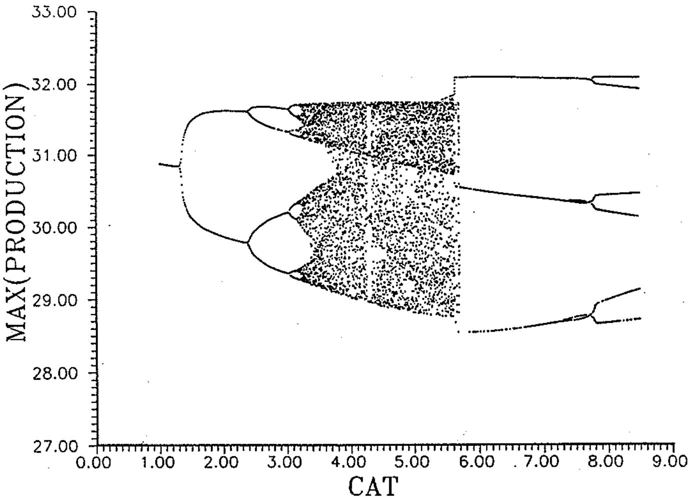

Figure 7. Bifurcation diagram for the commodity market model.

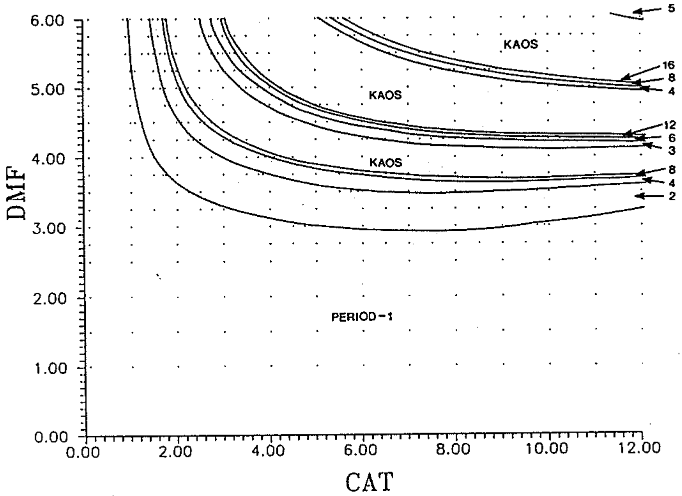

Figure 8. Distribution of stationary solutions for various values of the capacity adjustment time and the depreciation multiplier.

3. The Bacterial Operon

As discussed in the introduction to this paper, it becomes increasingly clear that many biological control systems operate in an unstable mode. There may be a number of advantages associated with such operation, and attempts to understand these are presently at the center of our interests. At this place, however, it suffices to note that many technical control systems including those of ordinary refrigerators, freezers, airconditioners, oil burners, etc. are constructed to operate in a bistable mode where they switch between an on state and an off state.

As an example of an oscillatory bacterial control system, Bliss et al. (3) have studied the tryptophan operon in E. coli. Experimentally they have observed that the gene expression from this operon becomes unstable if the normal feedback inhibition is reduced, and that self-sustained oscillations with a period of the order of 30 min result. These oscillations can be observed both in the intracellular concentration of tryptophan and in the concentration of the enzyme anthranilate synthetase which is involved in control of the tryptophan production.

Figure 9 shows a flow diagram for the model suggested by Bliss et al. (3). mRNA is produced by transcription of the bacterial DNA molecule. It is assumed that this process requires approximately 30 sec, and that the produced mRNA molecules have a lifetime of 60 sec. The concentration of mRNA molecules controls the production of the enzyme anthranilate synthetase through a sigmoidal relation. This implies that there is a threshold concentraton of mRNA below which very little enzyme production takes place, a control region in which the enzyme production increases significantly with the RNA concentration, and a saturation region in which changes in the RNA concentration have little influence on the enzyme production. The translation process is assumed to take 30 sec, and the produced enzyme molecules are assumed to have a lifetime of 500 sec.

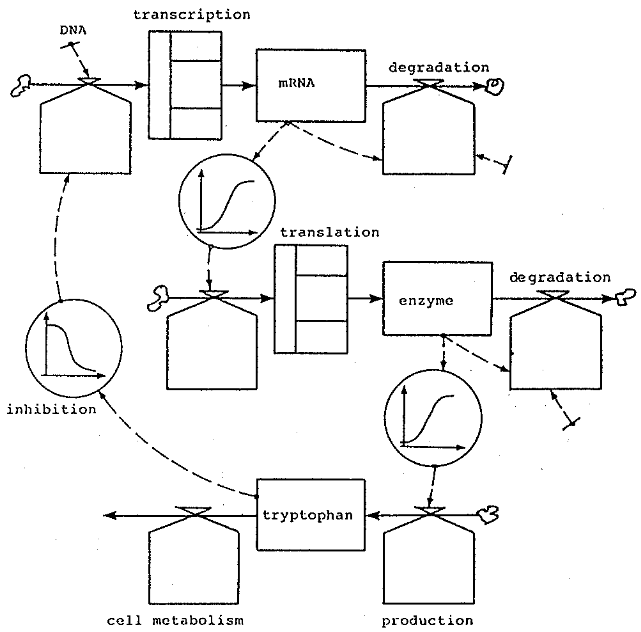

Figure 9. Flow diagram for the tryptophan operon.

The concentration of anthranilate synthetase controls the production of tryptophan, an amino acid which plays an important role for the cell metabolism. Also in this case, the control is via a sigmoidal relation. An advantage of this type of cascaded system is that a considerable amplification can be attained such that a few mRNA molecules can give rise to a significant number of tryptophan molecules.

Similar cascaded systems are found in human hormonal regulation. The production of testosteron (the male sex hormone), for instance, is regulated by the production of luteinizing hormone in the pituitary gland, and this production is again controlled by the production of luteinizing hormone releasing hormone in hypothalamus. The endocrine gland has a tendency to oversecrete its hormone. Because of this tendency, the releasing hormone exerts more and more of its control effect on the target cells in the pituitary gland which also overproduce. To restrain the hormone production, the cells at the final stage of the cascade usually produce a factor which feeds back to the first stage and causes the cells here to decrease their secretion Likewise, for our enzymatic control system in figure 9, the end product feeds back to the rate of mRNA transcription to reduce the transcription rate when the concentration of tryptophan becomes high enough. However, due to the involved time delays and phase shifts, the negative feedback regulation may become unstable and produce self-sustained oscillations.

The modification that we have introduced is to assume that besides the feedback from the concentration of tryptophan to the rate of RNA-transcription there is a parallel feedback from the concentration of anthranilate synthetase. Such a nested feedback is also characteristic of many biological systems. With this additional loop, the interaction between mRNA and anthranilate synthetase can become unstable all by itself. If started close to equilibrium this will produce an expanding oscillation in the enzyme concentration. However, when this concentration becomes sufficiently high an increased production of tryptophan is triggered. This activates the second negative feedback loop which brings the system back towards its equilibrium point. With this modification, the operon model produces the same types of behavior as the modified commodity market model: a cascade of period-doubling bifurcations leading to a chaotic attractor of Rössler type.

4. Conclusion

It has been the purpose of the present paper to illustrate how easily well-established economic and biological models can be modified to produce bifurcation cascades and Rössler like attractors. At the same time we have emphasized that endogeneously generated chaos is likely to occur in many biological systems. We have no reason to assume that managerial or macroeconomic systems should be more stable.

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