

## CHAOS STUDY IN SYSTEM DYNAMICS

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### ABSTRACT

Based on our study of synergetics and dissipative structure theory, by means of mathematics, in terms of the viewpoint of system dynamics, this paper concentrates on the study of chaos in system dynamics: we analyze the nature of chaos phenomenon and the characteristics of system dynamics, put forward the viewpoint that chaos testing should be included in model testing of S.D.; we investigate several necessary conditions of chaos; we create a model to question a famous sufficient condition of chaos; then, we shed some light on the way toward which chaos will occur. At last, we successfully apply our theoretic study to a standard nuclear spin generator model.

### 1. CHAOS AND SYSTEM DYNAMICS

System dynamics is a powerful tool to study the complex social and economic system. This complexity results from nonlinearity, high-order and multy loop of systems. One character of such complexity is that, though the system is deterministic, at certain point, its behavior is like stochastic. Apart from this, the system is extremely sensitive to the initial value. This strong sensitivity makes two trajectories which are very close in the beginning become quite different from each other in time. That is to say, the system will show chaotic behavior.

To a nonlinear dynamicist, a chaos system is one on which long-term predication of the system's state is impossible because the omnipresent uncertainty in determining its initial state and because the error of initial value grows exponentially fast in time.

So, it is of great significance to study chaos characteristics of S. D. models in order to make the model be more reliable.

Fortunately, System Dynamics has paid lots of attention to model testing. So, we recommend that chaos testing be included in sensitivity analysis while we do model testing.

Now, let us see the mechanics of chaos.

First, we classify all macro states in the world into Table 1.

Table 1.

state type	in-between state	final state
deterministic	stable	a. equilibrium b. periodical c. quasiperiodical
	unstable	d. tend to limitlessness e. sometimes limited sometimes limitless f. always limited
stochastic		

What classical theories study are the stable processes, namely, process a, b, and c; process d and e are divergent; process "f" is what we call chaos:

Def. 1. A state which is always limited and which is not equilibrium, not periodical, and not quasiperiodical is named chaos.

From the above discussion, we easily get:

Theorem 1. unstable, aperiodical, bounded and extremely sensitive to initial value are the necessary conditions of chaos.

In the following, we will concrete theorem 1.

For generality and transfer ability, this paper focuses on system (1):

$$\begin{aligned} dX/dt &= P(X, Y, Z) \\ dY/dt &= Q(X, Y, Z) \\ dZ/dt &= R(X, Y, Z) \end{aligned} \quad (1)$$

Set  $q = (X, Y, Z)$ , then, system (1) becomes

$$dq/dt = N(q) \quad (2)$$

## 2. BOUNDED

Def. 2. If existing a constant  $M$ ,  $q(t)$  is the solution of system (2), we have:

$$\limsup_{t \rightarrow \infty} q(t) < M \quad (3)$$

Then, we call system (2) is dissipative ([1]).

Theorem 2. Suppose  $q(t) = (X(t), Y(t), Z(t))$  is the solution of system (1), if  $(q(t), dq(t)/dt) < 0$ , then, system (1) is dissipative.

Proof: take

$$S \cdot S = X \cdot X + Y \cdot Y + Z \cdot Z \quad (4)$$

then  $(q(t), dq(t)/dt) < 0$

$$\begin{aligned} \langle \implies \rangle X \cdot dX/dt + Y \cdot dY/dt + Z \cdot dZ/dt &< 0 \\ \langle \implies \rangle S \cdot dS/dt < 0 \quad (\text{since } S > 0) \\ \langle \implies \rangle ds/dt < 0 \end{aligned}$$

So, system (1) is dissipative.

Example 1.

$$dX/dt = -bX + Y$$

$$\begin{aligned} dY/dt &= -X - bY(1 - kZ) \\ dZ/dt &= b[a(1 - Z) - kY*Y] \end{aligned} \quad (5)$$

where  $a, b, k > 0$ . We can verify  $(q, dq/dt) < 0$  (21).

Based on theorem 2, we can construct a kind of systems which are dissipative.

Theorem 3. In system (1), if P,Q,R satisfy:

- a)  $dq/dt = -A(q) + G(q) + H$
- b)  $(q, A(q)) > 0$
- c)  $(q, G(q)) = 0$
- d)  $G(0) = 0$

then  $(q, dq/dt) < 0$ . (When  $q$  is big enough)

### 3. UNSTABLE

In order to study unstableness, we introduce Lyapunov exponent.

Def. 3. Suppose  $dq(t)/dt = L(t)q(t)$  (6)

$L(t) = (a_{ij}(t))$ , if existing  $B > 0$   $\exists \sup |a_{ij}(t)| < B$

then, for the solution vactory  $q_i(t)$  of system (6), we define:

$$Z_i = \limsup_{t \rightarrow \infty} ((1/t) * \ln |q_i(t)|) \quad (7)$$

$Z_i$  are called generalized characteristic exponents. One special case of generalized characteristic exponent is so-called Lyapunov exponent.

Suppose  $q_0(t)$  is a solution of system (1),  $s_q(t)$  is a disturbance, then,

$$q(t) = q_0(t) + s_q(t) \quad (8)$$

Put (8) into (2):

$$dq_0(t)/dt + ds_q(t)/dt = N(q_0(t) + s_q(t))$$

$$\iff dq_0(t)/dt + ds_q(t)/dt = N(q_0(t)) + (dN(q_0(t)/dt)*s_q(t) + \alpha(s_q(t)))$$

$$\iff ds_q(t)/dt = [dN(q_0(t)/dt)*s_q(t)] \quad (9)$$

Def. 4. The generalized characteristics exponents of equation (9) are called Lyapunov exponents.

Haken in [3] asserts that, if the three Lyapunov exponents of system (1) are (+, 0, -), then, system is chaotic. In our study, we find this assertion is not correct.

Theorem 4. If  $q_0(t) = q_0$  then lyapunov exponents are the eigenvalues of  $N(q_0)$ .

Proof: Since  $q_0(t) = q_0$ ,  $N(q_0)$  is a constant matrix. So, equation (9) is a linear differential equation.

Suppose  $A_1, A_2, A_3$ , are the three eigenvalues of equation  $/AE - N(q_0)/ = 0$ ,  $V_1, V_2, V_3$  are relative eigenvectors. since  $A_i \exp(A_i t) V_i = N(q_0) \exp(A_i t) V_i \iff N(q_0) V_i = A_i V_i \iff (N(q_0) - A_i E) V_i = 0$

then,  $\exp(A_1 t) V_1, \exp(A_2 t) V_2, \exp(A_3 t) V_3$  are solutions of (9).

From Def. 3,

$$B_i = \limsup_{t \rightarrow \infty} \{ (1/t) * \ln |\exp(A_i t) V_i| \}$$

$$\limsup_{t \rightarrow \infty} \{ (A_i t)/t + \ln |V_i| / t \} = A_i$$

So, Lyapunov exponents = eigenvalues.

where  $|C|$  is the absolute value of  $C$ .

Example 2

$$\begin{aligned} dX/dt &= -bX + Y \\ dY/dt &= -X -bY + bkYZ \\ dZ/dt &= b - cZ - bkY*Y \end{aligned} \quad (10)$$

$b, c, k > 0$ . We have proved that system (10) is dissipative.

Theorem 5.  $C_1 = (0, 0, b/c)$ ,  $C_{2,3} = (\pm(1/b*b*k)*SQRT[(b*b*b*k-c(1+b*b))/b], \pm(1/b*k)*SQRT[(b*b*b*k-c(1+b*b))/b], (1+b*b)/b*b*k)$

are three equilibrium points of system (10).

Theorem 6. Take  $C = (b*b*b*k)/(1+b*b)$  (11)  
 $\implies A_1=0$  And  $A*A+(b-1/b+c)*A+(bc-c/b)=0, A_{2,3}=0.5[(1/b-b-c) \pm SQRT[(b-1/b+c)*(b-1/b+c)-4(bc-c/b)]]$   
 for  $b < 1, \implies A_2*A_3 < 0$  then, Lyapunov exponents of system (10) are  $(1/b-b, 0, -c) = (+, 0, -)$ . So, system (10) satisfies Haken's conditions. It should be chaos. But our study finds that system (10) has periodical solutions and it is not sensitive to initial values ([2]).

#### 4. APPLICATION

Example 2 shows that it is of little possibility to be chaotic directly at equilibrium points. Chaos is a dynamic process. The probable way toward which chaos may occur is:

a) change parameters to make equilibrium points be limited cycles.

b) go on changing parameters to let limited cycles bifurcate into cycle 2,4,8 .....

c) at last, go to chaos.

In step a, we have:

Theorem 7. Suppose  $W$  is a open set,  $0$  belongs to  $W$ ,  $W$  is included in  $R^*R^*R$ ,

$f: W*(-a_0, a_0) \implies R^*R^*R$   
 $dX/dt = f(x, a)$  (12)

$f(0, a) = 0$ , for any "a"

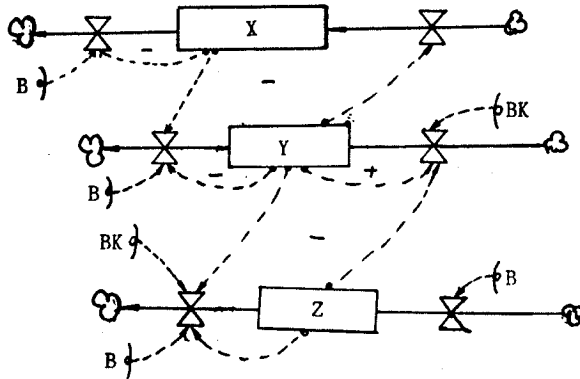
The eigenvalues of  $Df(0, 0)$  are  $+ib(0), -ib(0), S(0), S(0) < 0$ ,

$b(0) > 0$ , when  $a=0$ ,  $(0, 0, 0)$  is stable,  $a > 0$ .

$(0,0,0)$  lost stable.

Then, for small enough  $a$ , equation (12) has a stable closed trajectory near  $(0, 0, 0)$ .

Go along this road, we delve into a nuclear spin generator model. Example 3. (system 13)



This is a three levels, seven loops, nonlinear S.D. model. It comes from nuclear physics ([4]).

Theorem 8. Example 3 is dissipative.

Theorem 9. when  $b = 1$ ,  $k = 22$ , example 3 is chaos.

Our result are:

1. In example 1, we have proved that system (5) is dissipative.
2.  $C1 = (0, 0, 1)$ ;  $C2,3 = (\pm \text{SQR}[(k-1)*b*b-1]/k*b*b, \pm \text{SQR}[(k-1)*b*b-1]/k*b*b, (1+b*b)/k*b*b)$   
are the equilibrium points of system (5).
3. At point  $C=(0,0,1)$

$$M = \begin{pmatrix} -b & 1 & 0 \\ -1 & -b(1-k) & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$\implies A1 = -b < 0, A2,3 = 0.5[b(k-2) \pm \text{SQR}(b*b*k*k-4)]$$

$A1, A2, A3$  are Lyapunov exponents, or eigenvalues.

a)  $k < 2$ ,  $(k-1)*b*b - 1 < 0 \implies \text{Re}(A_i) < 0$ ,  $i = 1, 2, 3$

At this,  $C1$  is stable,  $C2,3$  have not meaning yet.

b)  $k < 2$ ,  $(k-1) - 1 > 0 \implies C$  have got meaning.  $A2*A3 < 0$ .

Proof: notice  $\text{SQR}(b*b*k*k-4) > /b(k-2)/$

$$\iff (k-1)*b*b-1 > 0$$

c)  $k > 2$ ,  $(k-1)*b*b - 1 < 0 \implies A2, A3 > 0$ .

d)  $k > 2$ ,  $(k-1)*b*b - 1 > 0 \implies A2*A3 < 0$ .

Under the condition of  $(k-1)*b*b - 1 > 0$ ,  $C2,3$  have meaning; the eigenvalue of  $C1$  are  $(-, +, -)$ ,  $(-, -, +)$ .

4. At  $C2,3$ ,

$$M = \begin{pmatrix} -b & 1 & 0 \\ -1 & 1/b & y1 \\ 0 & y2 & -b \end{pmatrix}$$

$$y1 = +\text{SQR}(\quad)$$

$$y2 = -2\text{SQR}(\quad)$$

$$\implies u1 = -b, u2,3 = 1/2\{(1/b - b) \pm \text{SQR}[(b-1/b)(b-1/b) - 8*(k-1)*b*b-1]\}$$

According to theorem 6, take  $b = 1$ , then

$$u2,3 = \pm \text{SQR}2*(\text{SQR}(k-2))i, \quad (k > 2)$$

At this time, at  $C1, A2,3 < 0$ ; at  $C2,3$ ,  $u2,3 = \pm \{\text{SQR}[2(k-2)]\}i$

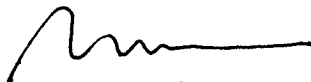
5. At  $b = 1$ ,

$$dA/db = \{[4(k-1)]/\text{SQR}[2(k-2)]\}$$

Therefore, this is a bifurcation point. So, go on.

Based on the above analysis, we simulated system (13):

a)  $b = 0.7, k = 1.5 \implies$



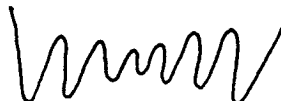
b)  $b = 2/3, k = 3.0 \implies$



c)  $b = 4.0, k = 1.1 \implies$



d)  $b = 0.5, k = 6.0 \implies$



.....  
When  $b = 1$ ,  $k = 22 \implies$  Chaos.  
In practice, people do find that the nuclear spin generator system will show irregular behavior. But they can not explain this phenomenon. Now our chaos study successfully shed light on it.

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