MODE–LOCKING AND CHAOS IN A PERIODICALLY DRIVEN MODEL OF THE ECONOMIC LONG WAVE

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Abstract

We have investigated the complex dynamic phenomena, which arise when the economic long wave model is perturbed by a sinusoidal variation in the orders for capital to the goods sector. This modulation represents a coupling to more short term oscillatory modes in the macroeconomic system. As the period of the external forcing is changed, a devil's staircase of frequency–locked oscillations develops. For higher amplitudes of the perturbing signal, period–doubling bifurcations, simultaneously existing periodic solutions, and deterministic chaos can be observed. The distribution of modes is determined as a function of the frequency and amplitude of the external signal. The phase diagram reveals characteristic bumps on the Arnol'd tongues, where they approach each other. The Lyapunov exponents are calculated, and the influence of noise is discussed in terms of the lock–in time for the periodic solutions.

Introduction

Macroeconomic systems are known to support a variety of different modes including the short term business cycle with a period of 3 – 7 years, the 15 – 25 year construction or Kuznets cycle, and the Kondratieff or economic long wave with a period of 45 – 60 years. If the macroeconomic system was completely linear, these modes could evolve independently of one another, and their underlying causes could be studied separately. However, economic time series from many different sources provide clear evidence for nonlinear interactions. It is well known, for instance, that tanker rates are particularly sensitive to variations in the demand for oil shipment during periods with high capacity utilization. During other periods, when surplus capacity exists, the tanker rates remain low and nearly constant.

Nonlinear phenomena arising from the interaction between short term business cycles and the economic long wave are also well documented. In particular, it has been found that the business cycle grows in amplitude towards the peak of the long wave (Sterman 1985 a). A similar picture has been observed both for the development of real wages, and for the variation of the Dow Jones index. In fact, the short term business cycle seemed almost to vanish during the initial growth phase of the post war expansion (Bronfenbrenner 1969). Since then it has reappeared and reached significant amplitudes. Moreover, the collapse of the long wave upswing appears to be triggered by a downswing of the business cycle.

In the same way, it has been suggested that long wave collapses are preceeded by a year or two by downturns in the building cycle (Long 1940). This does not imply that the turning points of the long wave are produced by more or less accidental coincidences between the business cycle and construction cycle variations. Rather, these and similar observations indicate that the various modes of the macroeconomic system interact with one another and adjust their periods such that each long wave spans a full number of Kuznets cycles, and each Kuznets cycle a full number of business cycles.

The occurrence of this type of entrainment between economic cycles of different periodicities was suggested early on by Schumpeter (1939). It has also been suggested by Forrester (1977) that entrainment can account for the uniqueness of the economic cycles. Oscillatory tendencies of similar periodicity in different parts of the economy are drawn together to form a single mode, and each of these modes are separated from the next by a wide enough margin to avoid entrainment at the same period. Hence, the economy exibits a number of clearly distinguishable modes rather than cycles of a broad spectrum. Apparently, however, these suggestions have never been carried over into a more formal analysis. Deterministic chaos arising from the interaction between different macroeconomic modes has been reported by Rasmussen et al. (1985) for a simplified version of the long wave model, and by Lorenz for a multisector business cycle model (1987 a and b). However, deterministic chaos is only one out of a great variety of complex dynamic modes, that can occur in such systems.

In the present work we have subjected the economic long wave model, developed by Sterman (1985 b), to a sinusoidal variation in the demand for capital to the goods sector, representing in this way the interaction between the Kondratieff wave and more short term macroeconomic cycles. As the period of the external signal is changed, a devil's staircase of frequency–locked oscillations develops. For higher amplitudes of the perturbing signal, period–doubling bifurcations, simultaneously existing periodic solutions, and deterministic chaos are observed. The distribution of modes as a function of the frequency and amplitude of the external signal shows a complicated Arnol'd tongue structure with characteristic bumps, located where the main tongues approach each other. To further characterize the different types of bifurcations that occur in the system, we have calculated its Lyapunov exponents.

The Model

The economic long wave model to be analysed below has been described in detail by Sterman (1985 b), who has also provided a complete list of the simulation equations. The model describes the flow of capital in the capital sector of an industrialized economy from its initial ordering to production, acquisition, application and discard. Because of their interference with the production of capital units for the capital sector, both ordering and production of capital units for the goods sector are also accounted for. At the core of the model is a positive feedback arising from capital self–ordering, i.e., from the fact that the capital sector depends on its own output to expand its production capacity. Because of this positive feedback, the model is inherently unstable: Even if the model is started in equilibrium, the slightest disturbance will generate an expanding oscillation. Well away from equilibrium, the behavior is confined by nonlinear restrictions associated, for instance, with capacity utilization and capital ordering. Without external forcing, the model thus exhibits a typical limit cycle oscillation with a period of 45–60 years. For the base case parameters considered here, the period is 47 years.

The state variables in the model are the stock of capital in the capital producing sector (KC), and the supply lines of unfilled orders for capital originating in both the capital sector itself (KLS) and in the consumer goods sector (GSL). Capital ordering from the goods sector (GCO) is assumed to be exogeneously determined. In the original version

(Sterman 1985 b), GCO was taken to be constant (GCO = 10^{12} capital units/year). In the present investigation we have superimposed a sinusoidal variation to represent the effect of more short term fluctuations in the economy. The relative amplitude of the sinusoidal drive is denoted A, and its period PER. The purpose of our work is to study how changes in PER and A influence the behavior of the long wave model, particularly how the long wave is entrained by the external signal. While in the real economy, entrainment is the result of mutual adjustments of the various cyclic modes to each other, our formulation clearly neglects the reaction of the long wave upon the periodicity of the more short term economic fluctuations. As a result, entrainment in the model is probably less pronounced than in real life.

Figure 1 shows a typical simulation result obtained with the long wave model without external forcing. We have here plotted the variation in production, production capacity, and backlog of orders for capital over a period of 150 years. This backlog equals the sum of the two supply lines KSL and GSL. The model explains the long wave in terms of subsequent expansions and contractions of the capital sector, as it strives to adjust its production capacity to the demand for capital. Once a capital expansion gets under way, self–reinforcing processes sustain it, until production finally catches up with orders, orders begin to fall, and excess capital is built up. At this point, the loops reverse. A reduction in orders further reduces investment demand, leading to contraction of the rate of production. Capital production hereafter remains below the level required for replacements and long term equilibrium, until the excess capacity has been fully depreciated. Due to the long lifetimes of capital, this may take a decade or two. When the excess capacity is finally got rid of, orders for capital rise again and trigger the next upswing.

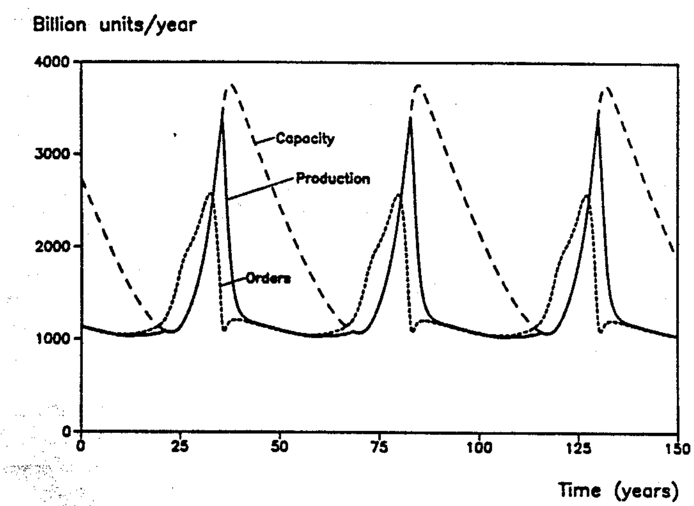

Figure 1. Base case result obtained by simulating the long wave model without external forcing. The three curves show the backlog of orders for capital (dotted curve), the rate of production of capital equipment (full line), and the capital sector production capacity (broken curve), respectively. The period of the wave is 47 years.

In the original version, the ordering policy is such that investments in the capital sector come to a complete stop during the contraction phase of the long wave. This occurs because the negative pressure generated by the existing excess capacity completely dominates the pressures to replace discards and to maintain adequate supply lines. As a result, the state variable KSL becomes virtually zero. This behavior does not seem to be quite realistic for a macroeconomic system with many independent firms ordering capital. In addition, it gives rise to unnecessary numerical problems in calculations of the Lyapunov exponents. In the present version we have therefore limited the negative investment pressure generated by the excess production capacity such as to allow a certain, though small, ordering of capital to the capital sector in the contraction periods. This explains the small differences between the results obtained in the base case simulation of the present paper (figure 1) and the corresponding results in the original publication.

Mode–Locking and Quasi–Periodic Behavior

For nonlinear systems the principle of superposition does not apply. In the presence of a periodic disturbance, the economic long wave therefore cannot exist as an independent mode, but it will adjust its behavior in accordance with the period and the amplitude of the external forcing. An interesting feature of this adjustment is that it tends to lock the two oscillations into an overall periodic motion. This is obtained when the oscillations have commensurate periods such that the long wave completes precisely q cycles each time the external forcing completes p cycles, where p and q are integers. Thus, if the model is perturbed by a signal which has a period different from, but relatively close to, the undisturbed Kondratieff period, the interaction between the two modes may cause the internally generated long wave to adjust its period, until the modes oscillate synchroneously. Similarly, if the period of the external signal is close to the fraction 1 : n of the undisturbed Kondratieff period, the model tends to adjust its internal period such that the long wave precisely completes 1 cycle each time the external signal completes n cycles.

As an example of this type of entrainment, figure 2 shows the results obtained when the model is perturbed by a 20 % (A = 0.20) sinusoidal modulation of the orders for capital to the goods sector (GCO). The period of the external signal is PER = 22.2 years, corresponding to a typical Kuznets cycle. Relative to the undisturbed simulation (figure 1), the long wave has increased its period by close to 40 % so as to accomodate precisely 3 periods of the external signal. Moreover, within the interval 19.9 years < PER < 24.8 years, a change in the period of the external signal will cause a precisely proportional shift in the period of the long wave such that the 1 : 3 entrainment is maintained. If PER is reduced or increased beyond these limits, sudden qualitative changes in the behavior of the model occur.

A clear illustration of the periodic nature of the mode–locked solution can be obtained by plotting phase–space projections of the stationary behavior, i.e., the behavior exhibited by the system, when all transients have died out. Figure 2b shows such a projection corresponding to the temporal variation depicted in figure 2a. We have here plotted simultaneous values of the capital sector capital (KC) and the goods sector capital orders (GCO) over a large number of subsequent long wave oscillations. The horizontal axis thus represents the external drive, and the vertical axis the response of the model. Inspection of the figure shows how the production capital in the capital sector builds up and decays precisely once for each 3 swings of the external signal.

Figure 3 shows the results obtained with the same amplitude of the external signal (A = 0.20), but with a modulation period of PER = 4.6 years. In this case, which could represent interaction between the economic long wave and the ordinary business cycle, we find a 1 : 10 entrainment. The long wave thus performs precisely one oscillation each

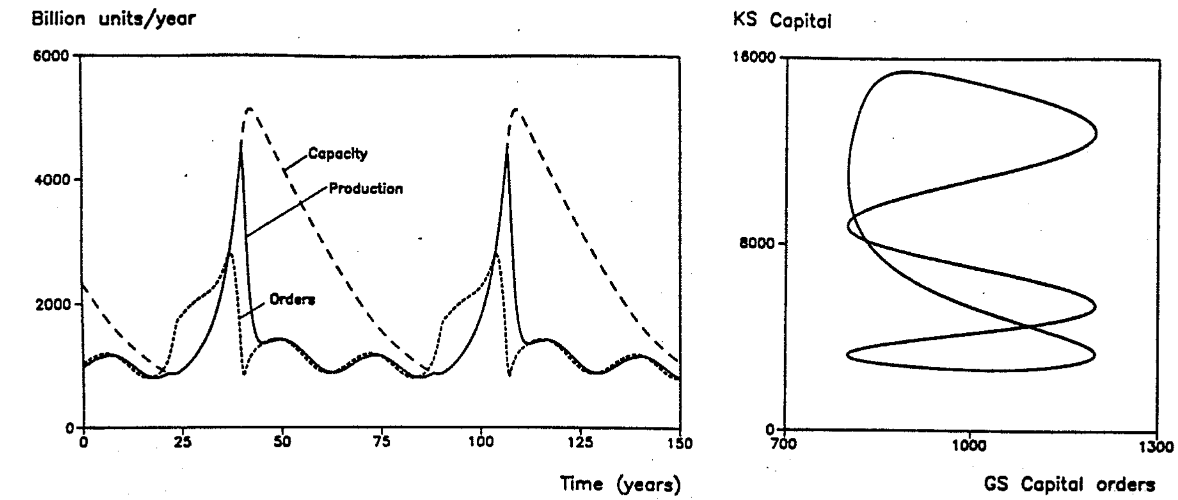

Figure 2. Simulation results obtained with a 20% periodic modulation of the orders for capital to the goods sector. The period of the external signal is PER = 22.2 years, corresponding to the period of a typical Kuznets cycle. The internally generated long wave has adjusted its period by almost 40% so as to accomodate precisely 3 Kuznets cycles. Figure 2b shows a phase space projection where we have plotted simultaneous values of the capital sector capital and the goods sector orders for capital.

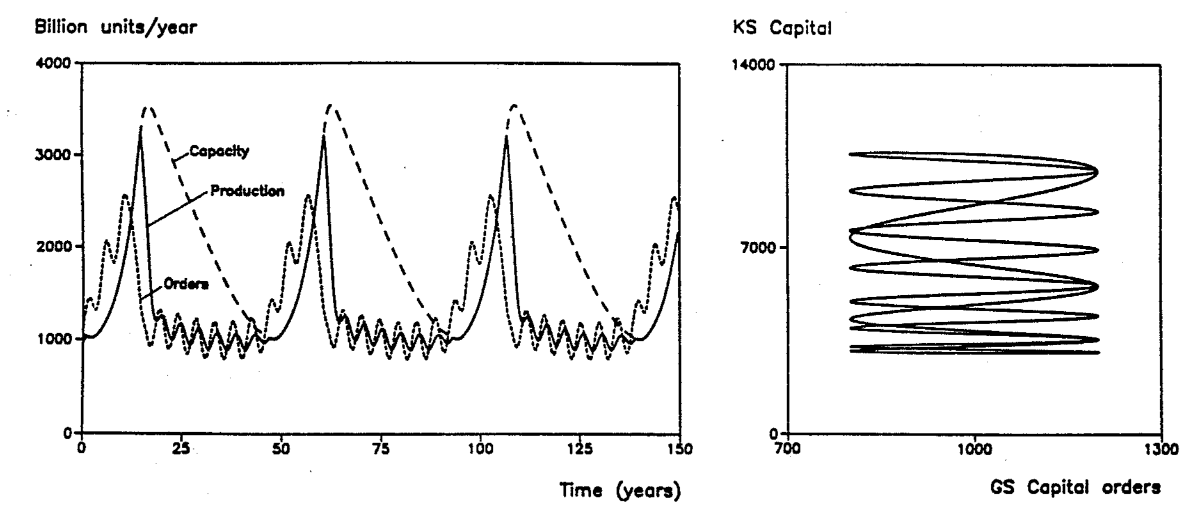

Figure 3. Simulation results obtained with a 4.6 year sinusoidal variation in the orders for capital to the goods sector. This modulation corresponds to a typical business cycle. To mode–lock with the external drive, the long wave has adjusted its period from 47 to 46 years. Figure 3b illustrates the periodic nature of the mode–locked solution.

time the business cycle completes 10 oscillations. This 1 : 10 mode–locked solution exists in the interval from PER = 4.47 years to PER = 4.70 years. Neighboring to the interval with 1 : 10 mode–locking we find intervals with 1 : 9 and 1 : 11 entrainment.

The interval in which a particular mode–locking occurs is a measure of the strength of the nonlinear interactions in the model. The interval therefore tends to widen with increasing amplitude of the modulating signal. The interval also depends upon the winding number, i.e., upon the ratio of the periods of the two interacting modes. Entrainment between modes with simple winding numbers and with winding numbers of the order of 1 is more pronounced than entrainment between modes with more complicated period ratios. 1 : 1 and 1 : 3 entrainment thus occurs over a wider range of PER than does, for instance, 1: 10 or 4 : 9 entrainment.

To illustrate the variety of different behaviors which can result from relatively weak pertubations of the long wave model, the following figures show the simulation results obtained with PER = 29.5 years (figure 4) and PER = 34.6 years (figure 5), respectively. The amplitude of the external signal has now been reduced to A = 0.05. Both figures show the temporal variation over an extended simulation period (600 years) together with the corresponding phase–space projection. The first simulation shows a 2 : 3 mode–locked solution in which the long wave peaks in production capital alternate between a high value and a somewhat lower value. Only after 2 complete long wave cycles (and 3 cycles of the perturbing signal) does the model repeat itself. Although convincing data are hard to find, there seems to be some evidence for such an alternation between high and low long wave peaks in the real economy.

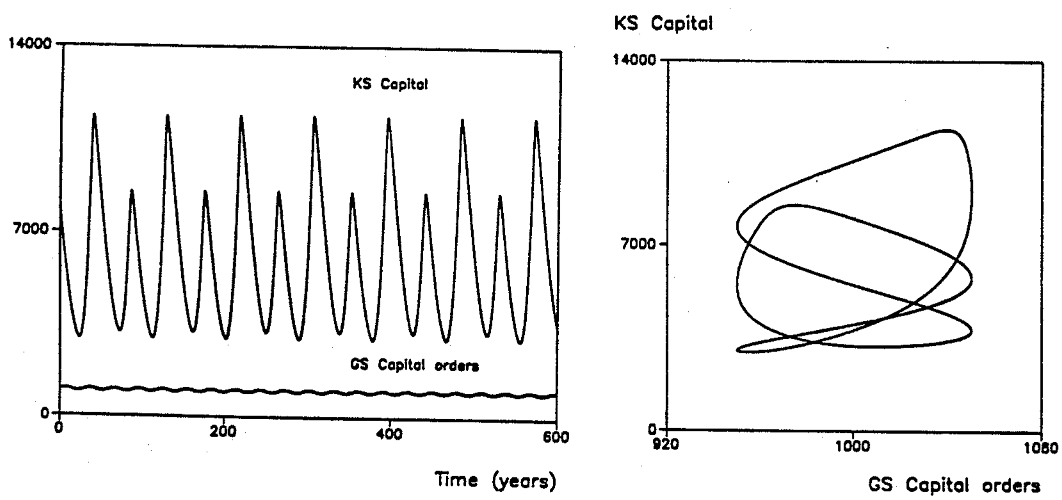

Figure 4. Temporal variation (a) and phase plot (b) for the 2 : 3 mode–locked solution obtained for PER = 29.5 years. The amplitude of the external forcing has now been reduced to A = 0.05. The long wave is seen to alternate between high and low peaks, and only after 2 complete long wave cycles (and 3 periods of the external signal) does the model repeat itself.

Figure 5 shows the 3 : 4 mode–locked solution existing for PER = 34.6 years. The model now performs 3 long wave cycles for each 4 cycles of the external signal. This type of entrainment only occurs in a relatively small interval for PER.

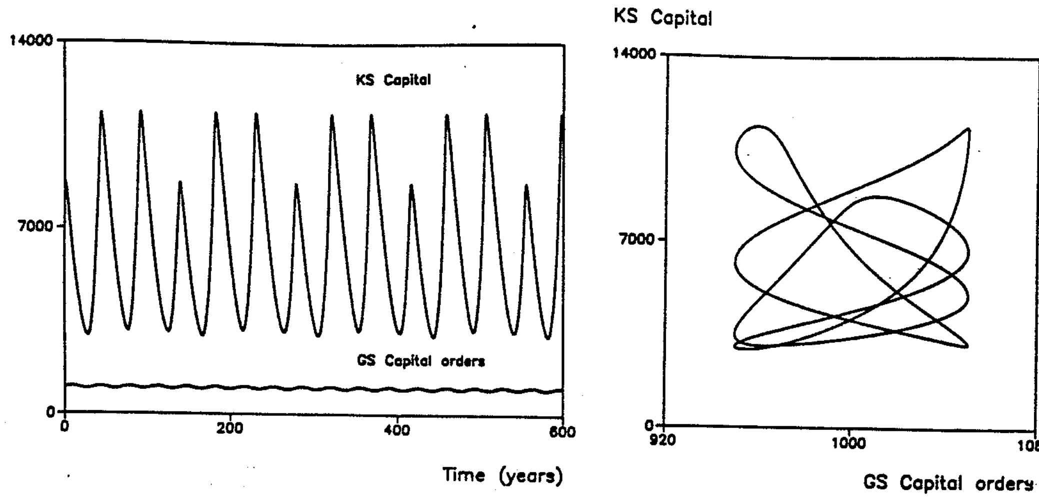

Figure 5. Temporal variation (a) and phase plot (b) for the 3 : 4 mode–locked solution which exists for with PER = 34.6 years and A = 0.05. In this mode the system performs precisely 3 long wave cycles for each 4 cycles of the external signal.

The model entrains at all rational winding numbers. For A > 0.025, the intervals of some of these solutions may overlap the intervals of other solutions. In these cases, several periodic solutions can exist simultaneously. The initial conditions will then determine which of the solutions the trajectory approaches. Between the intervals with mode–locked behavior, quasi–periodic and chaotic solutions can be observed. These are types of behavior in which the model never repeats itself, but continues to find new ways in phase–space. Quasi–periodic solutions, which occur for A < 0.025, are distinguished from deterministic chaos by their lack of sensitivity to the initial conditions. This again is reflected in the value of the largest Lyapunov exponent. Usually, the phase–space projection of a quasi–periodic solution also shows a much more orderly behavior than that of a chaotic solution.

A more complete picture of the entrainment process is obtained by plotting the observed mode–locking ratio as a function of the forcing period. Figure 6 shows an example of such a construction. The period of the external signal has here been varied from 2 to 60 years while keeping the amplitude constant at A = 0.05. The figure shows a series of 1 : n mode–locked solutions. Between these solutions, solutions with other commensurate wave periods are observed. In the region from PER = 29 years to PER = 37 years, we thus find intervals with 2 : 3, 3 : 4, 4 : 5, and 6 : 7 entrainment. For A = 0.025 we have identified 3 : 8, 2 : 5, 7 : 17, 3 : 7, 4 : 9, and 6 : 13 mode–locking between the regions with 1 : 2 and 1 : 3 entrainment.

By refining the calculations one can continue to find more and more resonances covering narrower and narrower intervals. At least for small values of A, i.e., below the critical line, where the mode–locked intervals start to overlap, the phenomenon has a self–similar structure, which causes it to repeat itself ad infinitum on a smaller and smaller scale. In practice, the finer details will be washed out by noise, that is, the random exogeneous events which continuously bombard the economy will not allow the trajectory to settle down in the neighborhood of one of the more complicated solutions. Simpler examples of mode–locking such as, for instance, 1 : 3 and 1 : 4 are quite likely to be observed in the real economy, however.

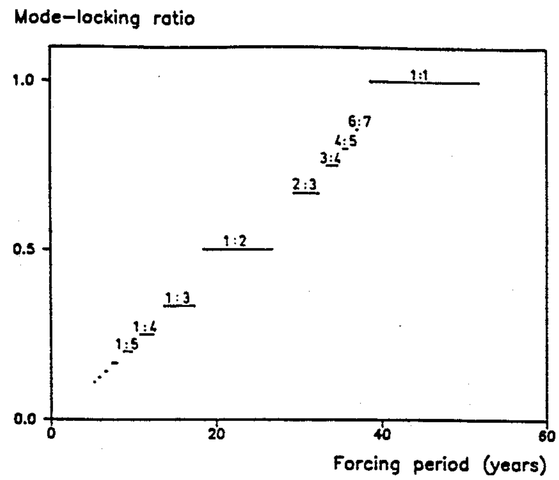

Figure 6. Intervals of the forcing period in which some of the main mode–locked solutions can be observed. Between these solutions, solutions with other commensurate winding numbers exist. This structure is refered to as a devil's staircase.

The structure exhibited in figure 6 is known as a devil's staircase. This structure has an universal character, which transcends the nature of the system (Jensen et al. 1983). Thus, essentially the same devil's staircase can be observed in the behavior of paced nerve cells (Colding–Joergensen 1983), periodically stimulated heart cells (Glass et al. 1986), and coupled thermostatically controlled radiators (Togeby et al. 1988).

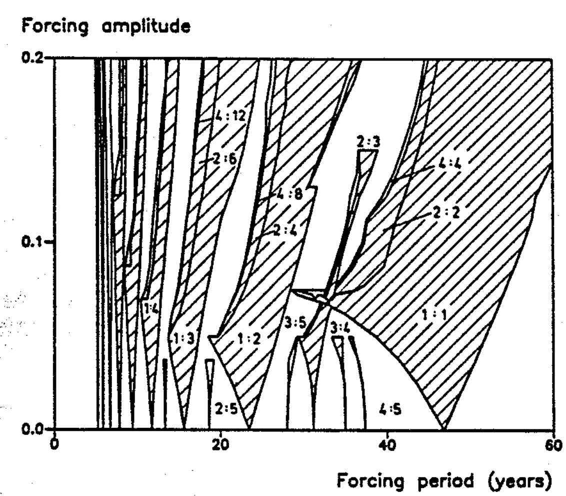

Figure 7. Phase diagram showing the regions of some of the main mode–locked solutions. These regions are refered to as Arnol'd tongues. Note the bumps on the tongues where they approach one another. Above the bumps, period–doubling bifurcations occur along the edges of the tongues.

Phase–Diagram

If the amplitude of the driving signal is changed, the intervals of entrainment will also change. An overview of this variation is provided by the phase–diagram in figure 7, which shows the range and position of some of the principal mode–locked zones as a function of A. These zones are commonly referred to as Arnol'd tongues (Jensen et al. 1984). For A = 0 there can, of course, be no entrainment at all. As A is increased, however, wider and wider intervals of mode–locked behavior start to develop. As long as A is still relatively small (A < 0.025), quasi–periodic behavior can be observed between the tongues. This type of behavior corresponds to irrational values for the winding number. The quasi–periodic trajectory thus winds around a torus without ever returning to the same point.

The widths of the tongues cannot continue to grow, however. At a certain point they will start to overlap, and quasi–periodic behavior then ceases to exist. This occurs at approximately A = 0.025. Above this critical value, the trajectory is either periodic or chaotic. Deterministic chaos is found to arise both via frustration (Jensen et al. 1984), and via period–doubling bifurcations. In frustrated chaos the trajectory switches at random back and forth between two or more periodic solutions. Figure 8 shows a typical example of chaotic behavior in the model. Here, we have plotted the temporal variation of the capital sector capital over a period of 1600 years. The period and amplitude of the perturbing signal are PER = 16.1 years and A = 0.20, respectively.

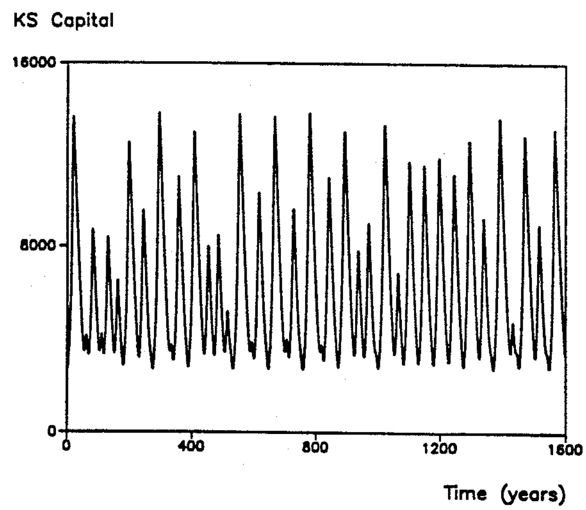

Figure 8. Temporal variations of the capital sector capital for an external pertubation of period PER = 16.1 years and amplitude A = 0.20. With this modulation the model shows deterministic chaos.

The Arnol'd tongue diagram in figure 7 has a number of interesting pecularities. In particular, we note the existence of characteristic bumps on the low period side of the primary tongues, where they approach one another. These are similar to the bumps observed by Cumming and Lindsay (1987) for a driven nonlinear electronic oscillator. For the 1 : 2 tongue there is also a bump on the high period side for A ≅ 0.12. The occurrence of these bumps indicates that the model is at variance with the universality theory for coupled resonances in dissipative systems (Jensen et al. 1983). Above the bumps, period–doubling bifurcations occur along the edges of the tongues. In contrast, the sine–circle map, from which the universality theory is derived, predicts a root–like structure of mode–converting bifurcations within each of the tongues.

As an example of the period–doubling behavior, figure 9 shows the cascade of bifurcations by which the 1 : 3 mode–locked solution is transformed into a 2 : 6, a 4 : 12, and an 8 : 24 solution as the period of the drive is increased from PER = 16.8 years to PER = 17.81 years while maintaining A = 0.20.

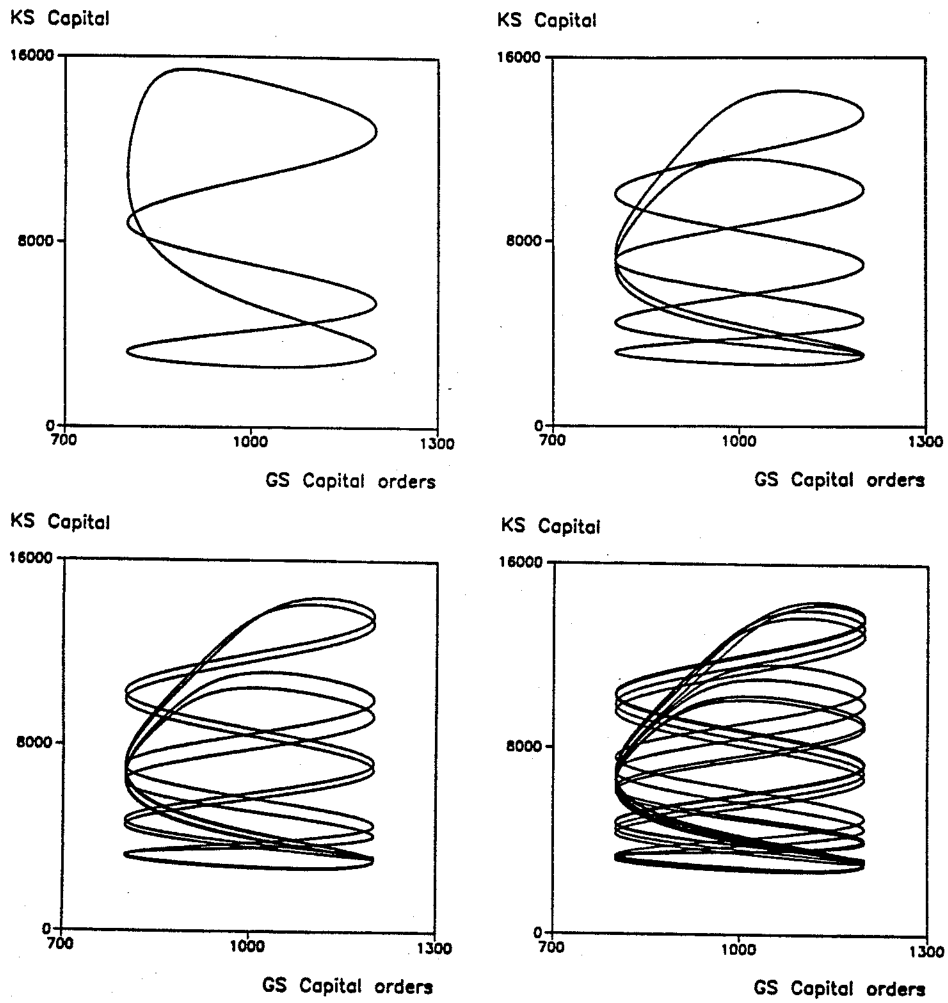

Figure 9. Phase plots illustrating the cascade of period–doubling bifurcations by which the 1 : 3 mode existing for PER = 22.2 years is transformed into a 2 : 6 mode (PER = 18.6 years), a 4 : 12 mode (PER = 18.0 years), and an 8 : 24 mode (PER = 17.81 years). In all simulations, the amplitude of the perturbing signal is A = 0.20.

In certain regions of the phase diagram (figure 7), several periodic solutions coexist, and the initial conditions determine which of these solutions the system chooses. This is, for instance, the case in the region around PER = 32 years and A = 0.07, where the 1 : 1 and the 2 : 3 tongues cross. In such regions, the model exhibits fractal basin boundaries, i.e., the initial conditions which lead to one periodic solution and those which lead to another, coexisting solution are separated by a fractal borderline. In certain regions of the phase–diagram we have found 6 or more coexisting solutions.

Lyapunov Exponents and Lock–In Time

Among the tools which have been developed to characterize complex behavior in dissipative systems, the Lyapunov exponents are some of the most useful (Wolf 1986). These exponents, of which there are as many as there are state variables in the system, measure the long term average rates of divergence or convergence of nearby trajectories.

A positive value of the largest Lyapunov exponent signals sensitivity to the initial conditions and deterministic chaos. For a driven nonlinear system as considered in the present study, a negative value of the largest Lyapunov exponent indicates a periodic orbit, while a quasi–periodic orbit is characterized by a vanishing value of this exponent. Bifurcation points correspond to orbits of marginal stability, and are also characterized by vanishing values of the largest Lyapunov exponent. In a period–doubling bifurcation the largest Lyapunov exponent is negative on both sides of the bifurcation point (and precisely zero at the point). For a tangent bifurcation, the Lyapunov exponent changes sign in the bifurcation point, and in a Hopf–bifurcation from a periodic orbit to a two–torus, the largest Lyapunov exponent is zero on one side of the bifurcation point and negative on the other.

For a periodic orbit, the reciprocal, numerical value of the largest Lyapunov exponent measures the time scale over which transients die out and the system locks onto its cyclic motion. The lock–in time provides a good measure of the stability of the orbit towards (small scale) random exogeneous disturbances.

To calculate the Lyapunov exponents of our periodically driven long wave model we have utilized a method described by Wolf (1986). We have followed the temporal variation of three small vectors $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$ where P(t) is a point on the stationary trajectory, and A(t), B(t) and C(t) are close by points on neighboring trajectories. As time goes by, these vectors expand or contract as the points A, B and C approach or diverge from the stationary solution. To maintain the lengths of the vectors within a proper dynamical range, and to avoid problems with orientational collapse along one of the axes, we have periodically applied a Gram–Schmidt reorthonormalization procedure in which:

(i) without change of direction the first vector is renormalized to a length ϵ, which is small compared with the diameter of the attractor and large in terms of the numerical accuracy,

(ii) the second vector has its component along the first vector removed, and is then renormalized to the length ϵ, and

(iii) the third vector has its components along the first two vectors removed, and is thereafter renormalized.

It is here assumed that the vectors have been ordered from most rapidly to least rapidly growing. Given a set of linarly independent vectors, the Gram–Schmidt reorthonormalization provides a new set of orthonormal vectors while at the same time preserving the orientation of particular subspaces: the direction of the first vector, the direction of the plane expanded by the two first vectors, etc..

With this procedure the Lyapunov exponents λ_1, λ_2 and λ_3 can be obtained from long term limiting values of

$$\lambda_1 = \frac{1}{t} \sum_i \ln\{\text{length}(t_i)/\epsilon\} \quad (1)$$

$$\lambda_1 + \lambda_2 = \frac{1}{t} \sum_i \ln\{\text{area}(t_i)/\epsilon^2\} \quad (2)$$

and

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{t} \sum_i \ln\{\text{volume}(t_i)/\epsilon^3\} \quad (3)$$

Here

$$\text{length}(t_i) = | \overline{PA}(t_i) | \quad (4)$$

is the length of the first vector immediately before renormalization at time t_i,

$$\text{area}(t_i) = | \overline{PA}(t_i) \times \overline{PB}(t_i) | \quad (5)$$

is the area expanded by the first two vectors, and

$$\text{volume}(t_i) = | \overline{PA}(t_i) \cdot (\overline{PB}(t_i) \times \overline{PC}(t_i)) | \quad (6)$$

is the phase space volume expanded by all three vectors just before renormalization. Σ_i extends over all renormalizations performed in the time t. The limit of long times is necessary to obtain quantities that characterize the stationary behavior, independent on initial conditions.

Figure 10 shows the variation of the largest Lyapunov exponent for the long wave model as a function of the period of the external drive. The amplitude of this drive is A = 0.05. For comparison, we have also shown the devil's staircase already depicted in figure 6. It is seen how the steps on the staircase correspond to regions with negative values for the Lyapunov exponent. Particularly for low forcing periods, the Lyapunov exponent becomes positive between the steps. This indicates the occurrence of deterministic chaos.

Figure 11 shows the complete spectrum of Lyapunov exponents. As expected, λ_2 and λ_3 are always negative, and the sum $\lambda_1 + \lambda_2 + \lambda_3$, which measures the rate of contraction of phase space, is also negative. It is interesting to note that the second Lyapunov exponent λ_2 shows a significant variation with PER. This can be used as a help to identify the bifurcation points. We have found that this structure becomes even more pronounced for higher amplitudes of the driving signal. In the regions of the phase diagram where the model shows simultaneously existing periodic solutions, λ_2 performs characteristic jumps.

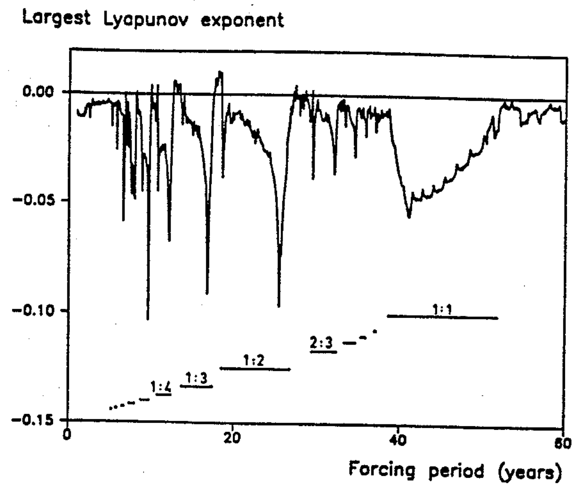

Figure 10. Variation of the largest Lyapunov exponent as a function of the period of the external drive. The amplitude of this drive is A = 0.05. To help identify regions in which mode–locked solutions exist, we have also plotted the corresponding devil's staircase. In intervals where the largest Lyapunov exponent is positive, the model shows chaotic behavior.

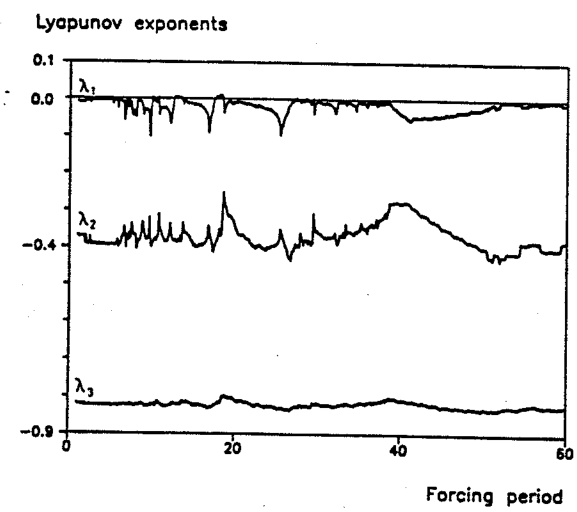

Figure 11. Complete spectrum of Lyapunov exponents as functions of the period of the external signal for A = 0.05. λ_2 shows a significant structure which can be used to identify the various bifurcation points.

Discussion

Macroeconomic systems distinguish themselves from most systems considered in the natural sciences by the prevalence of positive feedback loops. Well known examples include the accelerator and multiplier loops of ordinary Keynesian business cycle theories but these loops are only aggregate representations of a host of finer loops of which several have been mentioned in connection with our discussion of capital self–ordering. Other positive loops work through self–enhancement of growth expectations, amplification of capital requirements due to capital/labor substitution, and interactions via financial markets and innovation rates (Sterman 1985 b).

The presence of these positive feedback loops causes macroeconomic systems to exhibit a variety of different oscillatory modes with little, or sometimes even negative, damping. The economic long wave, for instance, appears to be generated as a self–sustained oscillation, and the short term business cycle, although stable in certain periods, appears to grow in amplitude in others. Such phenomena cannot be understood by means of linear or nearly–linear models. In particular, it is not possible to treat the various modes independently of one another, and entrainment processes are likely to play a significant role in many macroeconomic contexts.

It is important to realize that, when we speak about mode–locked solutions, we refer to the stationary behavior of the system after the transients have died out. If the system is excited by random external events, the picture becomes somewhat more complex. Each disturbance will knock the system out of its stationary orbit and, at least as long as the disturbance is sufficiently small, a new approach to the orbit will then begin. This approach will be characterized by a time constant, which is equal to one over the numerical value of the largest Lyapunov exponent. We may denote this time constant as the lock–in time.

For the long wave model, the time constant for entrainment into one of the primary model–locked solutions is of the order of 10–20 years, while lock–in times of the order of 20–50 years are found for the secondary solutions. In practice, this implies that only entrainment into the primary solutions can be observed in the economy. The system will never have time enough to settle down into one of the more complex mode–locked solutions, before a new external excitation again knocks it away from this orbit.

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