SUPPORT METHOD FOR OPTIMAL CONTROL OF LINEAR DYNAMICAL SYSTEM

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ABSTRACT. In this paper we construct a method for solving the problem of optimizing the maximal deviation of the real plan (or the trajectory, the technological process...) from some ideal one. The mathematical model of the dynamical system being considered is a system of linear differential equations with a control function. The method is based on some ideas of the so-called support method proposed by R. Gabasov and F.M. Kirillova.

After introducing support controls and establishing their relation with controllability of the system, we derive a criterion (which can be easily verified) for a support control to be optimal. Then we briefly describe an iteration for improving the existing control if it has not been optimal yet. Finally we present an illustrative example.

I. PROBLEM STATEMENT

Many situations arising from economic and technical practice can be formulated in the form of the following problem: Minimize the functional

$$L(y,u(.)) = \lambda = \max\{|d'x(t)-\alpha(t)|,\ t\in T=[0,t^*], \qquad (1)$$

with respect to the control $(y,u(.))$ and the phase trajectory $x(.)$ which satisfy the constraints:

$$\dot{x}(t)= A(t)x(t)+b(t)u(t)+c(t),\ t\in T; \qquad (2)$$

$$x(0)= x_0+ Gy,\ y_* \le y\le y^*; \qquad (3)$$

$$u_*\le u(t)\le u^*,\ t\in T;\quad Hx(t^*)= h. \qquad (4)$$

Here $x(t)$ is the state of the dynamical system (2), $x(t) \in R^n$, $A(t)$ is an $(n\times n)$-matrix, $b(t)\in R^n$, $c(t)\in R^n$, $u(t)$ is the scalar control function, $t\in T$, y is the control parameter, $y\in R^r$, G is an $(n\times r)$-matrix, H is an $(m\times n)$-matrix, rank $H= m\le n$, $x_0\in R^n$, $d\in R^n$, $h\in R^m$, $d'b(t)\neq 0$, $t\in T$, the symbol ' denotes transposition of vector or matrix, all the vectors are column vectors.

For example if $x_i(t)$ is the quantity of the i-th type product, $i = 1,2,\dots,n$, y is the control of the store at the initial time moment, $u(t)$ is the control of the technological process, $c(t)$ is the wear and tear, $\alpha(t)$ is the given in advance ideal plan of production at time moment t. Then $L(y,u(.))$ will be the maximal deviation from the given plan which we seek to minimize.

The problem (1)-(4), as we can see from the minimax and the absolute-value function in (1), is non-smooth and closely related to optimal control problems with state constraints. It is known that such class of problems is especially complicated to solve numerically and in recent years considerable efforts have been made to construct efficient methods and algorithms. In (Chan 1986; 1989) we investigated some simplified versions of the problem with $\alpha(t) \equiv \alpha$, $A(t) \equiv A$, $b(t) \equiv b$, $c(t) \equiv 0$, $G = 0$. Moreover, the results given in those papers were based on a strong non-singularity property of so-called support controls. In the present paper we shall consider another non-singularity property which seems to be the most natural one for the problem . This essential improvement became possible thanks to a new approach to prove the optimality criterion with using another type of control variations and applying the implicit function theorem. We exploit here mainly the ideas of Gabasov - Kirillova's support method which has been successfully used to solve a wide class of optimization problems (Gabasov and Kirillova 1980; 1984).

The plan of the paper is as follows. In Section 2 we introduce a support, a support control, and show the relation between existence of support and controllability, in a certain sense, of the system. An optimality criterion for a support control will be established in Section 3. Section 4 is devoted to description of a scheme for numerical solution of the problem. Finally, in Section 5 we present an illustrative example.

2. SUPPORT AND CONTROLLABILITY

In the following <u>an admissible control</u> is any pair $(y, u(.))$ where y is an r-vector, $u(.)$ is piece-wise continuous function, which satisfy all the constraints (2)-(4). We says that the admissible control $(y^0,u^0(.))$ is optimal iff $L(y^0,u^0(.)) \leq L(y,u(.))$ for every admissible control $(y,u(.))$.

Let $(y,u(.))$ be an admissible control and the maximum λ (see(1)) be attained on the set T_s^x of segments or isolated points: $T_s^x = \{ T_s^{xi} = [\tau_i,\tau^i] \subset T,\ \tau_i \leq \tau^i < \tau_{i+1},\ i \in I \}$ where I is a finite index set. Moreover, denote $I_* = \{i \in I: \min \{\omega_*(t),\ t \in T_s^{xi}\} = 0\}$, $I^* = \{i \in I: \min\{\omega^*(t),\ t \in T_s^{xi}\} = 0\}$, $T_{s*}^x =$

$\{T_s^{xi}, i \in I_*\}$, $T_s^{x*} = \{T_s^{xi}, i \in I^*\}$; $\omega(t) = \omega_*(t)$ for $t \in T_{s*}^x$, $= \omega^*(t)$ for $t \in T_s^{x*}$ where $\omega_*(t) = d'x(t) - \alpha(t) + \lambda$ (the lower deviation), $\omega^*(t) = \lambda - d'x(t) + \alpha(t)$, $t \in T$ (the upper deviation).

Definition 2.I (support controllability). The deviation $\omega(.)$ is said to be controllable on T_s^x provided that for any piece-wise-smooth function $z(.) = (z(t), t \in T_s^x)$ there is such variation Δy, $\Delta u(t)$, $t \in T$ and $\Delta\lambda$ that

$$d'\Delta x(t) + \rho_i \Delta\lambda = z(t), \quad t \in T_s^x, \tag{5}$$

$$H\Delta x(t^*) = 0 \tag{6}$$

where $\rho_i = 1$ if $i \in I_*$, -1 if $i \in I^*$,

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + b(t)\Delta u(t), \quad t \in T, \quad \Delta x(0) = G\Delta y. \tag{7}$$

On T_s^{xi} equation (5) is equivalent to the following conditions:

$$d'\Delta x(\tau_i) + \rho_i \Delta\lambda = z(\tau_i), \tag{8}$$

$$d'\Delta\dot{x}(t) = \dot{z}(t). \tag{9}$$

Combining (9) with (7), we obtain

$$\Delta u(t) = [\dot{z}(t) - d'A(t)\Delta x(t)]/d'b(t), \tag{I0}$$

$$\Delta\dot{x}(t) = A_s(t)\Delta x(t) + b_s(t)\dot{z}(t) \tag{II}$$

where $A_s(t) = [E - b(t)d'/d'b(t)]A(t)$, $b_s(t) = b(t)/d'b(t)$, $t \in T_s^x$, E is the identity diagonal matrix.

Denote by $F(t,\tau)$, $t \in T$, the solution of the matrix differential equation

$$dF(t,\tau)/d\tau = \begin{cases} -F(t,\tau)A(\tau), & \tau \in T_n^x = T \setminus T_s^x, \\ -F(t,\tau)A_s(\tau), & \tau \in T_s^x, \ \tau \leq t, \end{cases}$$

$$F(t,t) = E,$$

and put $F(t,\tau) = 0$ for $\tau > t$. Using the Cauchy formula for a solution $x(.)$ of the differential equation (7) (for $t \in T_n^x$) and (II)(for $t \in T_s^x$), we rewrite (8) and (6) into the form

$$d'F(\tau_i,0)G\Delta y + \int_{T_n^x} d'F(\tau_i,t)b(t)\Delta u(t)dt + \rho_i\Delta\lambda =$$

$$\gamma_i' z(\tau_I) + (\gamma^i)' z(\tau^I) + \int_{T_s^x} h_{fi}(\tau) z(\tau) d\tau, \qquad (I2)$$

$$HF(t^*,0)G\Delta y + \int_{T_n^x} HF(t^*,t)b(t)\Delta u(t)dt = F_c z(\tau_I) + F^c z(\tau^I) + \int_{T_s^x} h_c(\tau) z(\tau) d\tau \qquad (I3)$$

where all the vectors $z(\tau_I) = (z(\tau_1),\ldots,z(\tau_{|I|}))'$, $z(\tau^I) = (z(\tau^1),\ldots,z(\tau^{|I|}))'$ (the symbol $|I|$ stands for the number of elements of the set I), $\gamma_i = (\gamma_{i1},\ldots,\gamma_{i|I|})'$, $\gamma^i = (\gamma^i_1, \ldots,\gamma^i_{|I|})'$, the matrices F_c and F^c, and the functions $h_{fi}(\tau), \tau \in T_s^x$, $i \in I$; $h_c(\tau), \tau \in T_s^x$ can be computed from the data of the problem.

Thus we obtain a system of $|I|+m$ linear algebraic equations with respect to unknowns Δy, $\Delta u(t)$, $t \in T_n^x$ and $\Delta\lambda$. Controllability of the deviation $\omega(.)$ on T_s^x means that the system (I2)-(I3) is solvable for every value of the right-hand side. We are going now to establish a constructive condition for solvability of the system. To this end we define the $(|I|+m)\times r$ matrix

$$P^o = \begin{pmatrix} P^{os} \\ P^{oc} \end{pmatrix}$$

where

$$P^{os} = \begin{pmatrix} d'F(\tau_i,0)G \\ i \in I \end{pmatrix}, \quad P^{oc} = HF(t^*,0)G \text{ and let } p_k^o = \begin{pmatrix} p_k^{os} \\ p_k^{oc} \end{pmatrix}$$

be the k-th column of P^o, $k \in K = \{1,\ldots,r\}$.

<u>Theorem 2.I</u>. For controllability of the deviation $\omega(.)$ on T_s^x it is necessary and sufficient that there exist such columns p_k^o, $k \in K_S \subset K$, and points $t_j \in T_n^x$, $j \in J$, that one of two the following cases takes place:

a) (simple case): $|K_S| + |J| = |I| + m$ and $\det P_S \neq 0$ where the matrix

$$P_S = \begin{pmatrix} P_S^o & \begin{matrix} d'F(\tau_i, t_j)b(t_j),\ j \in J \\ i\ \ I \\ HF(t^*, t_j)b(t_j) \end{matrix} \end{pmatrix}, \quad P_S^o = (p_k^o, k \in K_S)$$

$$= \begin{pmatrix} P_S^{os} \\ P_S^{oc} \end{pmatrix} ;$$

b) (main case): $|K_S| + |J| = |I| + m - 1$ and $\det P_S \neq 0$ where the matrix

$$P_S = \begin{pmatrix} P_S^o & \begin{matrix} d'F(\tau_i, t_j)b(t_j),\ j \in J \\ i \in I \\ HF(t^*, t_j)b(t_j) \end{matrix} & \begin{matrix} \delta_i \\ \\ 0 \end{matrix} \end{pmatrix}$$

(Here one of the sets K_S and J may be empty).

To prove Theorem 2.I one has to note that if, for example, the case a) is realized, then there exists such a neighbourhood N_j of the point t_j, $j \in J$, that $\det \tilde{P}_S \neq 0$ where the matrix

$$\tilde{P}_S = \begin{pmatrix} P_S^o & \begin{matrix} \int_{N_j} d'F(\tau_i, t)b(t),\ j \in J \\ i \in I \\ \int_{N_j} HF(t^*, t)b(t)dt \end{matrix} \end{pmatrix}$$

Moreover, if $a_k(.)$, $k \in K$, are continuous linearly independent in T functions, then there exist such $|K|$ points $t_j \in T$, $j \in K$, that the matrix $(a_k(t_j))_{k,j}$ is non-singular.

In the main case, considering $\Delta u(t) \equiv \text{const}$ in a small interval $t[j]$ of t_j, $j \in J$, and then taking the limits as the length of the intervals converges to 0, formally we come to the following formula which plays an important part in the rest of the paper:

$$\Delta\lambda = \delta_N' \Delta y_N + \int_{T_n^u} \Delta_n(t)\Delta u(t)dt + \int_{T_s^x} \Delta_s(t)z(t)dt + \sum_{i \in I} [\delta_i z(\tau_i) + \delta^i z(\tau^i)] \tag{I4}$$

where δ_N, Δy_N, $\Delta_n(t)$, $t \in T_n^u$, $\Delta_s(t)$, $t \in T_s^x$, δ_i, δ^i, $i \in I$, can be computed from the data of the problem, q_1' is the last line of the matrix $Q_S = P_S^{-1}$.

Furthermore, denote $\tau_s^u = \{t_j, j \in J\}$, $T_s^u = \tau_s^u \cup T_s^x$, $T_n^u = T \setminus T_s^u$, $T_s = \{T_s^x, T_s^u\}$, $\tau_s = \tau_I \cup \tau_s^u$.

Definition 2.2. We say that the set $\{K_S, T_s\}$ is a support, the set T_s^x is a dynamical support, the set $\{K_S, \tau_s\}$ is a working support of the problem iff $\det P_S \neq 0$.

Definition 2.3. The pair $\{y, u(.); K_S, T_s\}$ where $(y, u(.))$ is an admissible control and $\{K_S, T_s\}$ is a support, is called a support control. We say that a support control $\{y, u(.); K_S, T_s\}$ is optimal, if the control $(y, u(.))$ is optimal, and non-singular, if it possesses two the following properties:

i) $y_{*k} < y_k < y_k^*$, $k \in K_S$; $u_* < u(t) < u^*$, $t \in T_s^x$;

$\omega_*(t) > 0$, $t \notin T_{s*}^x$, $\omega^*(t) > 0$, $t \notin T_s^{x*}$;

ii) every $t_j \in \tau_s^u$ is either a continuity point of the control function $u(.)$ and then $u_* < u(t_j) < u^*$ or a discontinuity point of $u(.)$.

We are ready now to establish the following

3. OPTIMALITY CRITERION

Theorem 3.I. The main case and the following relations are necessary and sufficient for optimality of a non-singular support control $\{y, u(.); K_S, T_s\}$:

$$\delta_{Nk} \begin{cases} \geq 0, & y_k = y_{k*} , \\ \leq 0, & y_k = y_k^* , \\ = 0, & y_{k*} < y_k < y_k^*, \quad k \in K_N ; \end{cases} \tag{I5}$$

$$\Delta_n(t)\begin{cases}\geqslant 0, & u(t)=u_*,\\ \leqslant 0, & u(t)=u^*,\\ =0, & u_*<u(t)<u^*, \quad t\in T_n^u;\end{cases} \tag{I6}$$

$$\Delta_s(t)\begin{cases}\geqslant 0, & \omega_*(t)=0,\\ =0, & \omega_*(t)>0,\end{cases} \quad \Delta_s(t)\begin{cases}\leqslant 0, & \omega^*(t)=0,\\ =0, & \omega^*(t)>0,\end{cases} \tag{I7}$$

$$t\in T_{s*}^x, \qquad t\in T_s^{x*},$$

$$\delta_i\begin{cases}\geqslant 0, & \omega_*(\tau_i)=0,\\ =0, & \omega_*(\tau_i)>0,\end{cases} \quad \delta^i\begin{cases}\geqslant 0, & \omega_*(\tau^i)=0,\\ =0, & \omega_*(\tau^i)>0,\end{cases}$$

$$i\in I_*,$$

$$\delta_i\begin{cases}\leqslant 0, & \omega^*(\tau_i)=0,\\ =0, & \omega^*(\tau_i)>0,\end{cases} \quad \delta^i\begin{cases}\leqslant 0, & \omega^*(\tau^i)=0,\\ =0, & \omega^*(\tau^i)>0,\end{cases} \tag{I8}$$

$$i\in I^*.$$

The proof of Theorem 3.I is not simple and needs more than one page. To prove the necessity one has to consider control variations of the following type:

$$\Delta u(t)=\begin{cases}\theta, \ t\in N(t_*);\\ \alpha_j \operatorname{sign}\varepsilon_j, \ t\in\{t_j, t_j+\varepsilon_j\}, \ j\in J_o, \ 0<\alpha_i< \\ \min\{u(t)-u_*, \ u^*-u(t), \ t\in[t_j-\varepsilon, t_j+\varepsilon]\}, \ |\varepsilon_j|<\varepsilon;\\ [u(t_j)-u(t_j+0)]\operatorname{sign}\varepsilon_j, \ t\in\{t_j, t_j+\varepsilon_j\}, j\in J_1;\\ 0 \text{ otherwise}, \ t\in T_n^x,\end{cases}$$

where t_* is a point of T_n^x where the relations (I5)-(I8) do not hold, $N(t_*)$ is a small neighbourhood of t_*, J_o is the set of continuity points of $u(.)$, $J_1 = J\setminus J_o$, ε is a given positive number, and to apply the implicit function theorem to the following $(|I|+m)$-dimensional vector-function

$$f'(\theta;\Delta y_k, k\in K_S, \varepsilon_j, j\in J, \Delta\lambda) = (P_S^o\,\Delta y_S)' +$$

$$\Big(\theta \int_{N(t_*)} d'F(\tau_i,t)b(t)dt + \sum_{j\in J_0} \int_{\{t_j,t_j+\varepsilon_j\}} d'F(\tau_i,t).$$

$$.b(t)dt\ \alpha_j \operatorname{sign}\varepsilon_j + \sum_{j\in J_1} \int_{\{t_j,t_j+\varepsilon_j\}} d'F(\tau_i,t)b(t)dt.$$

$$.[u(t_j)-u(t_j+0)]\operatorname{sign}\varepsilon_j + \varrho_i\Delta\lambda,\ i\in I,\ (\theta \int_{N(t_*)} HF(t^*,t).$$

$$.b(t)dt + \sum_{j\in J_0} \int_{\{t_j,t_j+\varepsilon_j\}} d'F(t^*,t)b(t)dt\ \alpha_j \operatorname{sign}\varepsilon_j$$

$$+ \sum_{j\in J_1} \int_{\{t_j,t_j+\varepsilon_j\}} d'F(t^*,t)b(t)dt[u(t_j)-u(t_j+0)] \operatorname{sign}$$

$\varepsilon_j)'\Big)$. Note that the sufficiency remains valid for any (not necessarily non-singular) support control.

4. CONTROL IMPROVEMENT

Assume that the support control $\{y,u(.);K_S,T_s\}$ being considered does not satisfy the optimality criterion yet. We are going now to describe briefly an iteration of the procedure of improving the support control.

Using the formula (I4), we seek to minimize $\Delta\lambda$ subject to the constraints

$$(7);\ y_* - y \leq \Delta y \leq y^* - y,\ \Delta y \in R^r;\ H\Delta x(t^*) = 0;$$
$$u_* - u(t) \leq \Delta u(t) \leq u^* - u(t),\ t\in T. \qquad (I9)$$

To this end we take positive parameters $\zeta_n, \zeta_s, \zeta, \ell$ and define sets

$$S_u = \{t\in T_n^u : |\Delta_n(t)| > \zeta_n\};\quad S_x^+ = \{t\in T_s^x : \Delta_s(t) > \zeta_s\},$$

$$S_x^- = \{t\in T_s^x :\ \Delta_s(t) < -\zeta_s\};\ N_{i1} = [\tau_i,\ \tau_i + 1),\ N_1^i =$$

$$(\tau^i - 1, \tau^i],\ N_1 = \bigcup_{i\in I} (N_{i1} \cup N_1^i).$$

Put

$$\Delta y_k = \theta_1(y_{*k} - y_k) \text{ if } \delta_{Nk} > 0,\ = \theta_1(y_k^* - y_k) \text{ if } \delta_{Nk} < 0,\ = 0 \text{ if } \delta_{Nk} = 0,\ k\in K_N; \qquad (20)$$

$$\Delta u(t) = -\theta_2[u(t) + \text{sign}\,\Delta_n(t)],\ t \in S_u, 0 \leq \theta_2 \leq 1; \quad (21)$$

$$d'\Delta x(t) + \Delta\lambda = -\theta_3 \omega_*(t),\ t \in S_x^+ \setminus N_1; \quad (22)$$

$$d'\Delta x(t) - \Delta\lambda = \theta_3 \omega^*(t),\ t \in S_x^- \setminus N_1,\ 0 \leq \theta_3 \leq 1. \quad (23)$$

Furthermore, assume that $N_{i1} \subset S_x^+$. Here are three possibilities:

1) $\int_{N_{i1}} \Delta_s(t)\omega_*(t)dt > \max\{\zeta, \delta_i \omega_*(\tau_i)\}$: In this case we introduce the condition (22) for N_{i1};

2) $\delta_i \omega_*(\tau_i) \geq \max\{\zeta, \int_{N_{i1}} \Delta_s(t)\omega_*(t)dt\}$: Introduce a new condition: $d'\Delta x(\tau_i) + \Delta\lambda = -\theta_4 \omega_*(\tau_i),\ 0 \leq \theta_4 \leq 1$;

3) $\zeta > \max\{\delta_i \omega_*(\tau_i), \int_{N_{i1}} \Delta_s(t)\omega_*(t)dt\}$: We take out the interval N_{i1} from the set N_1.

Analogously we consider every interval N_{i1}, N_1^i, $i \in I$. The resulting system of conditions will be numbered by (24).

The problem (1)-(4) formulated in terms of variations Δy, $\Delta u(t)$, $t \in T, \Delta\lambda$, with the additional constraints (20) -(24) is called <u>a continuous support problem</u>. Dividing the set $T \setminus (S_u \cup S_x^+ \cup S_x^- \cup N_1)$ into intervals $T_{(r)} = [t_{(r)}, t^{(r)}]$, $t^{(r)} - t_{(r)} \leq 1$, and putting $\Delta u(t) = v_r$, $t \in T_{(r)}$, $r \in R$, we arrive at <u>a discrete support problem</u> in the space of $\Delta y_S = (\Delta y_k, k \in K_S)$, v_r, $r \in R$, $\Delta\lambda$, $\theta_1, \dots, \theta_4$, which may be solved by the method of (Gabasov-Kirillova 1980). Thus transition to a new (improved) support control is completed.

5. EXAMPLE

To illustrate the method described in Secs. 2-4 we take the following example: Minimize the functional

$$L(y, u(.)) = \max\{|\ddot{x}(t)|,\ t \in T = [-1; 1]$$

subject to the constraints

$$\dddot{x}(t) - \dot{x}(t) = u(t),\ t \in T;\ x(0) = \dot{x}(0) = 0,\ \ddot{x}(0) = y,$$

$$1 \leq y \leq 2;\quad |u(t)| \leq 1,\ t \in T,\ (e+1)\dot{x}(1) - (e-1)\ddot{x}(1) = e-1.$$

Note that $\ddot{x}(0) = y \geqslant 1$. Therefore we have always $L(y,u(.)) \geqslant 1$. Moreover we shall consider only odd control functions $u(t) = -u(-t)$, $-1 \leqslant t < 0$.

We begin at the initial control $y = 1$, $u(.) \equiv 0$. For this control we have $\ddot{x}(t) = [\exp(t) + \exp(-t)]/2$. Hence $L(1;0) = (e+1/e)/2$.

The computations of Secs. 2-4 lead to the optimal control $y^o = 1$, $u^o(t) = -1$, $0 < t \leqslant 1$, $u^o(t) = 1$, $-1 \leqslant t \leqslant 0$ with $L(y^o, u^o(.)) = 1$.

REFERENCES

Gabasov, R. and Kirillova, F.M. 1980. Methods for Linear Programming. Part 3: Special Problems. Minsk: University Press.

Gabasov, R. and Kirillova, F.M. 1984. Constructive Methods of Optimization. Part 2: Control Problems. Minsk: University Press.

Nguyen Chan 1986. Direct Support Method for Solving a Continuous-Time Minimax Optimal Control Problem in Linear Ordinary Systems. Doklady of the Byelorussian Academy of Sciences. 6: 500-503.

Nguyen Chan 1989. Support Method for Solving a Linear Minimax Problem of Optimal Control. Optimization. 20, 4: 493-501.