AN ALTERNATIVE METHOD FOR POLICY SYNTHESIS OF A PRODUCTION-INVENTORY SYSTEM VIA MODAL CONTROL

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ABSTRACT

The feedback control concept is central to both modern control theory and system dynamics. Several attempts have been made to utilise modern control theory in developing some formalised procedures for system dynamics models. Some studies on the application of control theory to a simple production-inventory system have been reported. These studies were able to discover some important policy decisions to improve the behaviour of the system. This paper presents an alternative procedure based on modal control theory for designing useful policies for the production-inventory system. The results obtained in this study are generally similar to those obtained in earlier studies reported in the literature.

I. INTRODUCTION

System dynamics (Forrester, 1961; Forrester, 1968; Coyle, 1977) has been used successfully to model and simulate a class of production-inventory systems. However, the main shortcomings of the method are due to its unstructured, trial-and-error approach to realising acceptable policies. The number of policy alternatives is limited by the analyst's own experience and judgment, rather than by the attainment of objective criteria, such as stability of the system. As such, design policies for the system may vary in completeness among various analysts. It is thus necessary to develop some formalised procedures to design appropriate policies for the system. The feedback concept is fundamental to both modern control theory and system dynamics. One is able to draw a parallel between the two (Mohapatra, 1980). By transforming the system dynamic model into a mathematical form suitable for modern control theory application, one can utilise modern control theory to develop stable system behaviour. One study(Mohapatra and Sharma, 1985) uses modal control theory(Porter, 1972) to highlight the usefulness of such an approach to policy synthesis in a production inventory system (Coyle, 1977).

This paper presents an alternative method of synthesising policies in the production-inventory system by using eigenvalue assignment. The method is based on state feedback but it utilises a different eigenvalue assignment algorithm. Furthermore, the eigenvalues are assigned simultaneously rather than one by one as has been done in other studies. The approach is suitable for computer implementation. The policies obtained by the method presented in this paper are better than those propounded in the original study(Coyle, 1977) and generally similar to those obtained in the study mentioned above(Mohapatra and Sharma,1985).

II. EIGENVALUE ASSIGNMENT USING STATE FEEDBACK

The state feedback approach is well established and a large number of algorithms exist for the solution of this kind of problem. Instead of using algorithms that assign eigenvalues one at a time(as in Mohapatra and Sharma, 1985) an algorithm for synthesis of state feedback regulators by entire eigenstructure assignment(Porter and D'Azzo, 1977) should be more suitable. Because of simplicity and power of the algorithm, the full rank eigenvalue assignment method using state feedback is usedin this study. This method is also nicely suited for machine computation. Considering a linear time invariant continuous system described in state-space form as

$$X=AX+BU \quad (1)$$

where A,B are real matrices with dimension nxn, nxr respectively, the equations which describe the state feedback problem consists of equation (1) plus the relationship

$$U=-KX \quad (2)$$

where K is the feedback gain matrix. Combining equations (1) and (2) we obtain the resultant closed-loop system equation

$$X=[A-BK]X \quad (3)$$

Equation (3) indicates that the eigenvalues of the closed-loop state feedback system are the roots of

$$L(\lambda_i)=|\lambda_i I-(A-BK)|=0 \quad (4)$$

When the matrix pair (A,B) is completely controllable, it is possible to assign a set of desired eigenvalues λ_i's for i = 1, n to equation (4) by appropriately using a constant feedback matrix K (Wonham,1965). In order to synthesise the system with real physical flow, all elements of K must be real. The problem is to determine K such that equation (4) is satisfied for n specified values of λ_i. The method presented here for finding K is adapted from (Kimura,1975). Since equation (4) is true, there exists at least one non-zero vector w_i such that

$$(\lambda_i I-A+BK)w_i=0 \quad (5)$$

Rearranging equation (5)

$$(A-BK)w_i=\lambda_i w_i \quad (6)$$

Hence it is clear that w_i is an eigenvector of the closed-loop system matrix (A-BK) associated with the closed-loop eigenvalue. Rewriting equation (6)

$$(\lambda_i I-A)w_i=-BKw_i$$

or

$$[\lambda_i I-A|B]\begin{bmatrix} w_i \\ Kw_i \end{bmatrix}=[0] \quad (7)$$

The coefficient matrix in the above homogeneous equation has rank n for any value of λ_i, if the matrix pair (A,B) is completely controllable. Equation (7) is true as long as equation (4) is true for each specified eigenvalue in the assigned set of eigenvalues. Let the maximal set of linearly independent solution vectors for a given eigenvalue λ_i be called $U(\lambda_i)$. The columns of U constitute a basis for the null space of $[\lambda_i I-A|B]$. The solution of the homogeneous equation in equation (7) can be found in the literature. From equation (7) the solution of the eigenvectors can be partitioned into two parts as

$$U(\lambda_i)=\begin{bmatrix} w_1 & w_2 & w_3 \dots w_r \\ h_1 & h_2 & h_3 \dots h_r \end{bmatrix}=\begin{bmatrix} W(\lambda_i) \\ H(\lambda_i) \end{bmatrix} \quad (8)$$

where the substitution h=Kw has been made by collectively combining equation (8) into one equation for the set of assigned eigenvalues λ_i, i=1,...n. The relationship becomes

$$K[W(\lambda_1), W(\lambda_2), \dots W(\lambda_n)]=[H(\lambda_1), H(\lambda_2), \dots H(\lambda_n)] \quad (9)$$

Equation (9) by itself is overdetermined and cannot be solved directly for K. However, if the system is controllable a nonsingular nxn matrix of $w_j(\lambda_i)$'s can be determined by selecting n linearly independent columns from both sides of equation (9) and ignoring the remaining columns. In the selection process, one column from both sides of equation (9) must be used for each specified eigenvalue λ_i. Let the n columns selected from the left hand side be called G and the corresponding columns from the right hand side be called F. Then the following equation would result

$$KG=F \quad (10)$$

Therefore, the feedback gain matrix K can be solved using

$$K=FG^{-1} \quad (11)$$

The vectors which are arbitrarily selected to form matrix G must meet the requirement that its inverse exists (i.e. G is nonsingular). Also, any linear combination of columns from a given partition of W can be used as long as the same linear combination of H is used. A preliminary first cut screening for linearly dependent columns can be implemented into the selection process. This is done by performing the Gram Schmidt orthonormalisation process and checking the inner products of $W^T W$. Each pair of vectors w_i^T and w_j must satisfy the relationship $<w_i^T w_j>=\delta_{ij}$, δ_{ij} is the Kronecker delta. The vectors are linearly independent only if $\delta_{ij}=1$ when $i = j$ and $\delta_{ij}=0$ when $i \neq j$. In the above mentioned discussion, it was assumed that the set of desired eigenvalues to be assigned are distinct. However, if it is desired that repeated eigenvalues are to be assigned, then equation (6) has to be altered slightly. When one or more eigenvalues is a repeated root of the characteristic equation, a full set of n linearly independent eigenvectors may or may not exist. For each repeated eigenvalue λ_i with degeneracy q_i, there will be q_i eigenvectors and Jordan blocks associated with λ_i. If q_i is less than the multiplicity of λ_i, then generalised eigenvectors will be required. A generalised eigenvector or rank k is defined as a non zero vector satisfying

$$[A-\lambda_i I]^k x_k = 0 \quad \text{and} \quad [A-\lambda_i I]^{k-1} x_k \neq 0$$

The entire set of generalised eigenvectors is generated by the rule

$$x_{k-1} = [A-\lambda_i I]x_k$$
$$x_{k-2} = [A-\lambda_i I]x_{k-1}$$
$$\vdots \qquad \vdots$$
$$x_1 = [A-\lambda_i I]x_2$$
$$0 = [A-\lambda_i I]x_1$$

One method of finding the generalised eigenvector is by first finding the eigenvectors and then building up a chain of generalised eigenvectors from them. That is, first find all solutions of the homogeneous equation

$$[A-\lambda_i I]x_i = 0$$

for the repeated eigenvalue λ_i. For each x_i thus determined, we try to construct a generalised eigenvector using

$$[A-\lambda_i I]x_{i+1} = x_i$$

If the resultant x_{i+1} is linearly independent of all vectors found then it is a valid generalised eigenvector. If more generalised eigenvectors are needed we can proceed as follows:

$$[A-\lambda_i I]x_{i+2} = x_{i+1}$$

Therefore, if generalised eigenvectors are needed, equation (6) can be written as

$$[\lambda_i I - A | B] \begin{bmatrix} w_{g+1} \\ Kw_{g+1} \end{bmatrix} = w_g \qquad (12)$$

where w_g is the first generalised eigenvector.

Equation (12) is used in place of equation (7) when a certain eigenvalue is repeated more times than allowed by eigenvector considerations. Also an iterative searching algorithm of the form of a modified Gram-Schmidt orthonormalisation process is used on the composite matrix $[w(\lambda_1)|w(\lambda_2)|..|w(\lambda_n)]$ to guarantee the selection of an independent set.

III. NON-UNIQUENESS OF GAIN MATRIX K

From equation (4), the eigenvalues of a closed-loop system with state feedback are the roots of the characteristic polynomial

$$L(\lambda_i) = |(A-BK)-\lambda_i I| \qquad (13)$$

The eigenvalue placement then becomes the problem of finding a suitable feedback gain matrix K which can give specified roots for equation (13). (Brocket, 1965) has shown that for a single-input completely controllable system, there always exists a unique K for each specified set of eigenvalues. However, from the design procedures discussed previously, many different feedback gain matrices can be found by selecting various combinations of eigenvectors. For completely controllable open-loop multi-input system, the feedback gain matrix K is not unique for each set of specific roots of equation (13). It will be important to make the best use of these various possibilities in order to determine a gain matrix K which is easiest to realise in practice. We can, for example, choose the one which allows us

to assign the smallest coefficients to those states which are hardest to reconstruct or most affected by noise.
A system can be made stable by applying state feedback and assigning a suitable set of eigenvalues. The dynamic behaviour of a system largely depends on the position of its closed loop eigenvalues. The process of locating the eigenvalues of a closed loop system arbitrarily in the complex plane by using state feedback is known as eigenvalue assignment. However up to this point, what constitutes a desired set of eigenvalues is still undefined. This depends entirely on the performance criteria of the system such as stability, response time, sensitivity to disturbances and other system dynamics related to closed loop eigenvalue locations. Even, if these criteria are precisely specified, there is no simple solution to the problem. The control law expressed by the feedback control matrix K can be obtained through the eigenvalue assignment method used here. The feedback control matrix K, provides important guidelines which are used to improve the model structure and policies of the industrial model.

IV THE PRODUCTION-INVENTORY MODEL

The production-inventory system by (Coyle, 1977)was used by (Mohapatra and Sharma, 1985) to illustrate the use of modal control theory in the design of policy. This model is now taken as another reference example to show the simplicity, effectiveness and advantages gained in using the eigenvalue assignment method presented here. Comparisons are then made among the results obtained by (Coyle, 1977),(Mohapatra and Sharma, 1985) and those obtained here. One further advantage of using the production-inventory model is the low order which favours hand computation in the absence of a computer program.

The company has two departments, distribution and manufacturing. Distribution holds stock to meet sales and the stock is replenished from manufacturing. In manufacturing, the backlog of unfilled orders changes the production rate. Goods are sent to distribution after a delay. The model has three state or level variables,

PLA=pipeline content actual
OBL=order backlog
INV= inventory

Two rates, factory order rate (FOR) and production start rate (PSR) control the control the flow of material and are the policy variables policy variables in the system. Coyle's final model structure uses an an additional pure level variable, work in final assembly(WIFA) as well as the policy variable, delivery from factory(DFF). Coyle gave the following equations for the three policy variables:

$$\left.\begin{aligned} &\text{FOR.KL=ASR.K+}\frac{\text{(DINV.K-INV.K)}}{\text{TAIP}}+\frac{\text{(PLD.K-PLA.K)}}{\text{TAPL}} \\ &\text{PSR.KL=APL.K} \\ &\text{DFF.KL=AOR.K+}\frac{\text{(WIFA.K-DWIFA.K)}}{\text{TAWI}} \end{aligned}\right\} \quad (14)$$

The reduced basic model with delinked policy variables is used as the starting point for the eigenvalue assignment process. The information flows are taken out and and so are average sales rate (ASR), average order rate (AOR), average production level(APL) and auxiliary variables like desired inventory (DINV), required backlog (RBL) and indicated production level (IPL). The policy variables are now delinked and the order of the modelreduced to three instead of the original eight. The third- order production delay has also been assumed to be of order one.

V CONTROL POLICY DESIGN OF THE MODEL

Using the reduced model,the linear differential equations containing FOR, PSR, PLA, OBL and INV are:

$$\left.\begin{aligned} \dot{PLA} &= PSR-DFF \\ \dot{OBL} &= FOR-PSR \\ \dot{INV} &= DFF-SR \end{aligned}\right\} \quad (15)$$

The above state differential equations can be obtained through standard procedure,Mohapatra[3]). By regrouping the differential equations into vector matrix form as

$$\dot{X}=AX+BU+ED \quad (16)$$

where

$$A=\begin{bmatrix} -\frac{1}{PDEL} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{PDEL} & 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad E=\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$X=\begin{bmatrix} PLA \\ OBL \\ INV \end{bmatrix}, \quad U=\begin{bmatrix} FOR \\ PSR \end{bmatrix} \quad \text{and } D=[SR]$$

In the above equation, U is the vector of control variables and D is the exogeneous variable.

The following step by step algorithm based on the theory described above is applied here to illustrate the application of the eigenvalue assignment method to the reduced production-inventory model.

Step 1: The complete controllability of the reduced system model is checked. A general criteria used for doing this check is to form the controllability matrix

$$P=[B|AB|A^2B|\ldots|A^{n-1}B] \quad (17)$$

where A and B are the system matrices. The linear system is completely controllable if and only if the n x rn matrix P has rank n(Wonham, 1965).

For comparison purposes, the same numerical values for the eigenvalues as used by (Mohapatra and Sharma, 1985) are assumed, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = -2$.

Substituting PDEL=6 in the relationship for A we obtain

$$P=[B|AB|A^2B]$$

$$=\begin{bmatrix} 0 & 1 & 0 & -1/6 & 0 & 1/36 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/6 & 0 & -1/36 \end{bmatrix}$$

Thus P has rank of 3. Therefore, the system is completely controllable.

Step 2: Form the homogeneous equation

$$[\lambda_1 I-A|B]\begin{bmatrix} w_1 \\ Kw_1 \end{bmatrix}=[0]$$

To solve the homogeneous equation, we assume the dummy variables e_1, e_2, e_3, e_4 and e_5 and obtain the following equation:

$$\begin{bmatrix} (1/6+\lambda_1) & 0 & 0 & 0 & 1 \\ 0 & \lambda_1 & 0 & 1 & -1 \\ -1/6 & 0 & \lambda_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}=[0]$$

Solving the above equation, the following linearly independent eigenvectors are obtained:

$$\begin{bmatrix} 1 \\ 0 \\ \frac{1}{6\lambda_1} \\ -(\frac{1}{6}+\lambda_1) \\ -(\frac{1}{6}+\lambda_1) \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\lambda_1 \\ 0 \end{bmatrix}$$

However, there are three repeated eigenvalues and the degeneracy q_1 is two. Therefore. generalised eigenvectors are required. Using equation(13) and w_1 equal to the second eigenvector above we have,

$$\begin{bmatrix} (1/6+\lambda_1) & 0 & 0 & 0 & 1 \\ 0 & \lambda_1 & 0 & 1 & -1 \\ -1/6 & 0 & \lambda_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ \frac{1}{6\lambda_1} \end{bmatrix}$$

Solving for e_1, e_2, e_3, e_4, and e_5 the corresponding eigenvector is obtained as follows:

$$\begin{bmatrix} 0 \\ 1 \\ \dfrac{1}{6\lambda_i^2} \\ 1-\lambda_i \\ 1 \end{bmatrix}$$

Step 3: Form the composite matrix for all eigenvalues λ_i's. The composite matrix for this example is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1/12 & 0 & 1/24 \\ \hline 11/6 & 2 & 3 \\ 11/6 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \Omega \\ \hline \Phi \end{bmatrix}$$

Step 4 : Since there are only three linearly independent eigenvectors and three assigned eigenvalues,

$$G=\Omega \quad \text{and } F=\Phi.$$

Step 5: Solving the equation KG=F, we obtain

$$K=FG^{-1}=\begin{bmatrix} 11/6 & 2 & 3 \\ 11/6 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -24 \\ 2 & 0 & 24 \end{bmatrix}$$

$$=\begin{bmatrix} 23/6 & 2 & 24 \\ 23/6 & 0 & 24 \end{bmatrix}$$

Check: To check that the closed loop eigenvalues of (A-BK) are indeed the assigned eigenvalues, $\lambda_1=\lambda_2=\lambda_3=-2$, we have

$$(A-BK)=\begin{bmatrix} -4 & 0 & -24 \\ 0 & -2 & 0 \\ 1/6 & 0 & 0 \end{bmatrix}$$

which gives eigenvalue of -2 with multiplicity of 3 as assigned. Now we can write the equation for the control vector U as follows:

$$U=-KX=\begin{bmatrix} FOR \\ PSR \end{bmatrix}=\begin{bmatrix} -23/6 & -2 & -24 \\ -23/6 & 0 & -24 \end{bmatrix}\begin{bmatrix} PLA \\ OBL \\ INV \end{bmatrix}$$

Therefore,

$$\left.\begin{aligned} FOR(t) &= -(23/6)PLA(t)-2OBL(t)-24INV(t) \\ PSR(t) &= -(23/6)PLA(t)-24INV(t) \end{aligned}\right\} \quad (18)$$

VI. MODIFICATION OF THE ORIGINAL MODEL

The control policies obtained from the application of the eigenvalue assignment (equation (18)) give the expression for factory order rate, FOR and production start rate, PSR. They are in general agreement with the results of (Mohapatra and Sharma, 1985). The negative sign of the terms reflects the negative feedback.

The expression for FOR indicates a negative negative relationship with the level of pipeline delay, order backlog and inventory. The factory order rate, FOR, is the rate at which distribution orders goods from manufacturing, and feedback control is obtained by means of order backlog, OBL, and work in progress denoted by PLA. Manufacturing adjust their production against the backlog of unfilled orders and the amount of work in progress, PLA, so as to deliver the finished goods to distribution after a delay. These finished goods then become inventory, INV. The amount of actual pipeline content (or work-in-progress) PLA directly affects the inventory INV level in the transient period. Control is achieved by taking into account the large amount of goods inside PLA as well as OBL. In this way the system responds faster and more to changes in the exogenous variable, sales rate, SR.

In the expression for production start rate, PSR, the order backlog term is absent and PSR depends heavily on pipeline delay and inventory. The production start rate, PSR, is controlled by both PLA and INV. PSR is be cut back as the levels of inventory, INV and pipeline order, PLA rise. This information on the dependence of production start rate, PSR on pipeline order, PLA and on inventory, INV have also been derived by (Mohapatra and Sharma, 1985). It is shown here and also by (Mohapatra and Sharma, 1985) that new information sources can be derived through the use of modal control theory. However, it is not realistic to apply the values provided in K of equation (18) directly on a system dynamics model. By examining matrix K, some important facts may be observed. It is important that the factory order rate, FOR should contain the terms, pipeline content actual, PLA, order backlog, OBL, and inventory, INV as shown in equation (18).

The states used in equation (18) are departures or deviations of the variables from the desired values. Also, the initial values of each of the state variables is implicitly assumed to be zero when equation (18) was generated. However, the exogenous variable, sales rate, SR has an initial value of 100 units per week. Therefore, a realistic policy design for factory order rate, FOR should contain a term of average sales rate, ASR (To ensure initial steady state condition) and the terms showing deviation of the desired values for actual pipeline content, PLA, inventory, INV.

The equation for factory order rate, FOR is as follows:

$$FOR=ASR + \frac{(RBL-OBL)}{TABL} + \frac{(PLD-PLA)}{TAPL} + \frac{(DINV-INV)}{TAI} \quad (19)$$

and for production start rate, PSR we would have

$$PSR=ASR + \frac{(DINV-INV)}{TAIP} + \frac{(PLD-PLA)}{TALP} \quad (20)$$

where TACL, TAL, TAI, TAIP and and TALP are time constants.

For comparison purposes the values of the values of the time constants used here will be similar to that used by (Mohapatra and Sharma, 1985) and they are as follows: TABL=6 , TAI=TAPL=TALP=12 and TAIP=4. The design policies given by equations (19) and (20) are then used in lieu of the original rates given by (Coyle, 1977) as

$$FOR=ASR+\frac{(DINV-INV)}{TAI}+\frac{(PLD-PLA)}{TAPL}$$

and PSR=APL

The modified model is then simulated using DYNAMO.

VII. RESULTS AND DISCUSSIONS

The simulation results of the revised model reflecting the policies in equations (19) and (20) are similar to those. obtained by (Mohapatra and Sharma, 1985). The revised policies determined in this paper give better results in several aspects than those obtained in the original study(Coyle, 1977).

The fluctuation in INV is much smoother and it comes back to a steady state in a much shorter time than in Coyle's model. The inventory level does not go down as much as in the original model, and the order backlog decreases substantially. The revised model reacts much faster and settles down much earlier. There is also improvement in the variation of factory order rate, FOR. The variation is smoother without abrupt jump between time 10 weeks and 20 weeks. Similar improvements can also be noted in the production start rate, PSR. The variables settle down in almost the same time as Coyle's model.

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