Studies of a Model of Entrainment between Economic Cycles

Juan Hernández-Guerra. *

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Summary

This work carries out an analytic study of a two-sector model previously developed by Kampmann et al. (1993). This model explains why two capital producing sectors with initially different cyclical motions can entrain to only one cyclical mode. Some new insights into the motion of the model will be shown. All the introduced assumptions and results were tested by simulations.

1 Introduction

The model consists of ten ordinary differential equations including the following variables: K_{pq}, that is the capital stock produced by sector q located in sector p, B_p, that is the sector p's total orders backlog, and S_{pq}, that indicates the supply line of capital produced by sector q to sector p (p,q $\in \{1,2\}$). The two sectors are connected by two specific parameters. They are the difference between each sector's capital lifetime, $\Delta\tau$, and the dependence between sectors α, $0\leq \alpha \leq 1$.

The objective of our work is to study analytically the extreme cases of the model (α=0, α=1). Case α=0 is important because the system changes from a totally independent sectors to dependent ones. Then, possibly this point will define a bifurcation. Case α=1 is also relevant. In this situation, each sector's production depends on the capital produced by the other sector solely. It is observed by simulations (Kampmann et al. 1993, Haxholdt et al. 1994) that synchronization fills all the $(\alpha, \Delta\tau)$ phase portrait for high values of α. By means of the analytic

*Universidad de Las Palmas de Gran Canaria. Phone: (34) 28451803. E.mail:juanh@empresariales.ulpgc.es

treatment, we will show the cause of this phenomenon.

2 Analytic Study of the Model

Using similarities with Sterman's model (1985), we can consider the following equation:

$$B_p = S_{qp} + S_{pp} + S_{G_p} \quad ; \quad \forall p, q \in \{1,2\},\ p \neq q \tag{2.1}$$

where S_{G_p} is the supply line from sector p to goods producing sector. The equality has been verified by simulations run in the original model.

We will also take one sector as exogenous. So, we will reduce the number of the equations and emphasize essential properties in the model. This will be observed in each particular analysis.

After incorporating the new assumptions, the ten equations are transformed into: (Note the sector treated as exogenous using subindex i, and the sector treated as non exogenous using subindex j)

$$\begin{aligned} K'_{ji} &= x_i - x_i^e - \frac{K_{ji}}{\tau_i} \\ K'_{jj} &= x_j - x_j^e - \frac{K_{jj}}{\tau_j} \\ B'_i &= O_{ji} - (x_i - x_i^e) \\ B'_j &= O_{jj} - (x_j - x_j^e) \end{aligned} \tag{2.2}$$

where x_h^e (h=i or j) is the proportional part of sector h's production corresponding to the exogenous sectors. We will consider it as exogenous.

- *Independence between Sectors, $\alpha = 0$:*

Incorporating α=0 in 2.2, we obtain two groups of decoupled equations. One group includes the variables (B_j, K_{jj}) and the other group the variables (B_i, K_{ji}).

It is easy to observe that (B_j, K_{jj}) system achieves a period variable limit cycle depending on the parameters governing sector j. On the other hand, (B_i, K_{ji}) system includes only one globally asymptotically stable equilibrium point, that is $(\overline{B}_i, \overline{K}_{ji}) = (S_{e_i}, 0)$.

However, it can be proven that the equilibrium point is not hyperbolic, so the structural stability of the model with α=0 is not assured. We will go back on this item later.

- *Total Dependence between Sectors, ($\alpha = 1$):*

Forcing α=1, we obtain four no decoupled equations. The only equilibrium point in the system is $(\overline{B}_j, \overline{K}_{jj}, \overline{B}_i, \overline{K}_{ji}) = (S_{e_j}, 0, \delta_i(x_i^e + x_j^e \frac{\kappa}{\tau}), \frac{\kappa_{ji}}{\delta_j} S_{e_j})$. It can be proven that this equilibrium point is locally asymptotically stable if the following conditions over the parameters are verified:

$$\frac{\tau_j}{\delta_j} > \frac{\tau_j^K}{\tau_j^S} \quad ; \quad \frac{\tau_i}{\delta_i} > \frac{\tau_j^K}{\tau_j^S} \tag{2.3}$$

Realistic values of the parameters verify this condition (Sterman 1989).

In order to study the global motion of the system in this point we are going to analyze the groups of variables (B_j, K_{jj}) and (B_j, K_{ji}) separately. So, it is easy to prove that $(\overline{B}_j, \overline{K}_{jj}) = (S_{e_j}, 0)$ is the only equilibrium point in the first group and it is globally asymptotically stable. The other group admits only one equilibrium point, note $(\tilde{B}_i, \tilde{K}_{ji})$ (if $B_j = \overline{B}_j$, then $(\tilde{B}_i, \tilde{K}_{ji}) = (\overline{B}_i, \overline{K}_{ji})$). So, using some Lemmas in Theory of Dynamical System (compare Hirsch and Smale 1974), we can assert that the equilibrium point is globally asymptotically stable. Then, $(\overline{B}_j, \overline{K}_{jj}, \overline{B}_i, \overline{K}_{ji})$ is globally asymptotically stable.

It can also be proven that the motion of the model is qualitatively persistent under small perturbations in the parameters defining the system. So, when a small change of parameter α occurs ($\alpha < 1$, $\alpha \simeq 1$), only one equilibrium point of the new model can still find out. This point is close to $(\overline{B}_j, \overline{K}_{jj}, \overline{B}_i, \overline{K}_{ji})$ and globally asymptotically stable too.

In accordance with above, we can assert that when one sector is in equilibrium (exogenous), the other one will be necessarily in equilibrium. Thus it is shown that the motion of each sector pulls the motion of the other sector. That is the cause of why synchronization fills the $(\alpha, \Delta\tau)$ phase portrait for high values of α.

3 Discussion

The analytical study is a quite good tool to find out some properties that simulation can not do. Here the model was not easy to attack, however we can draw some new results from this paper.

First, we have delimited a region in the parametric space where the equilibrium point in case α=1 is asymptotically stable (equations 2.3). Moreover, starting from the equilibrium point when $\alpha = 0$ is not hyperbolic, a necessary condition of a bifurcation is obtained. Hence, we can support the idea that entrainment arise since any value of $\alpha > 0$. On the other hand, structural stability of the system in case α=1 lead us to infer that synchronization fills the $(\alpha, \Delta\tau)$ phase portrait for α no much less than one and any $\Delta\tau$.

The model can be extended including more sectors and some new economic linkages. There are some new results on this line (Kampmann 1996). To incorporate another macroeconomic linkages is our next goal. Then the use of simulated and, if it was feasible, analytical methods would help us to study the motion of the new model.

4 References

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